

Metastability for a stochastic dynamics with a parallel heat bath updating rule*

Emilio N.M. Cirillo¹ and Francesca R. Nardi²

¹Dipartimento Me. Mo. Mat., Università degli Studi di Roma “La Sapienza”,
via Antonio Scarpa 16, I-00161 Roma, Italy.
E-mail: cirillo@dmmm.uniroma1.it

²Eurandom, PO BOX 513, 5600MB, Eindhoven, NL.
E-mail: nardi@eurandom.tue.nl

Abstract

We consider the problem of metastability for a stochastic dynamics with a parallel updating rule with single spin rates equal to those of the heat bath for the Ising nearest neighbors interaction. We study the exit from the metastable phase, we describe the typical exit path and evaluate the exit time. We prove that the phenomenology of metastability is different from the one observed in the case of the serial implementation of the heat bath dynamics. In particular we prove that an intermediate chessboard phase appears during the excursion from the minus metastable phase toward the plus stable phase.

*AMS 1991 subject classification: 60K35; 82B43; 82C43; 82C80. Keywords and phrases: stochastic dynamics, probabilistic cellular automata, metastability, low temperature dynamics.

1. Introduction

Metastable states arise when a physical system is close to a first order phase transition. If the system is prepared in the metastable phase, it takes an extremely long time to reach equilibrium. In physical experiments it is seen that if the system is not suitably perturbed it remains forever in the metastable phase [PL].

A rigorous description of metastability cannot be formulated in terms of the standard equilibrium statistical mechanics: dynamical models must be considered [CGOV]. The case of the stochastic serial dynamics has been discussed, for instance, in [KO, NS1, S]: at each step of time one of the spins on the lattice is updated with rates satisfying the detailed balance condition. In this set-up it has been seen that starting from the wrong metastable phase, the time needed by the system to exit the metastable state, namely the *exit time*, is exponentially long in the inverse of the temperature. Moreover, the exit time is exactly the time needed to see a sufficiently large droplet, namely the *protocritical droplet*, of the stable phase in the metastable background. Hence, the equilibrium is achieved via the *nucleation* of such a protocritical seed.

It is rather natural to ask oneself in which sense these results depend on the dynamics. In this paper we consider a dynamics in which simultaneous spin flips [BCLS, C] are allowed: the single spin flipping rates are those corresponding to the two dimensional nearest neighbors Ising interaction. More precisely we study the metastable behavior of a Probabilistic Cellular Automaton [R, St] which is reversible with respect to a Gibbs measure derived by an Hamiltonian with four body interaction. We show that the exit path from the metastable phase to the equilibrium changes dramatically, with respect to the serial implementation of the heat bath dynamics, in particular the system visits an intermediate metastable phase before reaching the equilibrium. This is not surprising, indeed, as it will be pointed out throughout the paper, there exist many deep differences between the evolution of the system under a serial and a parallel dynamics.

We focus, now, on what we consider the most relevant novelty appearing in the study of metastability for parallel dynamics: in Glauber dynamics the system can jump between configurations differing at most for one spin, such pairs of configurations are called neighboring configurations. A connected domain is a subset of the configuration space such that for any pair of states it is possible to find a sequence of pairwise neighboring configurations of the domain joining the two states; the system, during its evolution, can visit the whole connected domain without exiting from the domain itself. In order to exit a connected domain, the system must necessarily cross its external boundary, that is the set of configurations not belonging to the domain, but having a nearest neighbor inside it. This sort of “continuity” property is the key property in estimating the exit time, that is in establishing the minmax between the metastable and the stable states, namely the minimal energy barrier bypassed by any path joining the metastable to the stable state.

Continuity is absent in the case of PCA’s: any configuration is connected to any other, a path joining the metastable to the stable state is an arbitrary sequence of configurations starting with minus one and ending with plus one. The lack of continuity forces us to develop techniques to estimate the energy cost of any direct jump from a subcritical to a supercritical configuration.

The paper is organized as follows: in Section 2 we define the model. In Section 3 we state our results: we first characterize the stable configurations (fixed points for the zero temperature dynamics, that is the typical droplets of the right phase plunged into the sea of the wrong phase); then we study the tendency to grow or to shrink of such droplets; finally, we construct the subset of the configuration space visited by the system in the metastable phase (description of the fluctuation around the metastable state) and, via a detailed description of the escape path, we estimate the exit time. In Sections 4 and 5 we, finally, prove the Theorems and the Propositions.

2. Definition of the model

In this section we define our model, namely a Probabilistic Cellular Automaton reversible with respect to a four body hamiltonian.

2.1. Preliminary definitions

Let Λ be a two-dimensional torus containing L^2 lattice sites, i.e., $\Lambda \subset \mathbb{Z}^2$ is a square containing L^2 points and having periodic boundary conditions. Let $d : (x, y) \in \Lambda \times \Lambda \rightarrow d(x, y) \in [0, +\infty)$ be the euclidean distance on the lattice Λ . For any $X, Y \subset \Lambda$ we define $d(X, Y) := \inf_{x \in X, y \in Y} d(x, y)$.

We say that $x, y \in \Lambda$ are *nearest neighbors* iff $d(x, y) = 1$. We say that the set $X \subset \Lambda$ is a *cluster* iff for any $x, y \in X$ there exist $x_1, \dots, x_k \in X$ such that $x_1 = x$, $x_k = y$ and for any $i = 1, \dots, k-1$ the two sites x_i and x_{i+1} are nearest neighbors.

Given two integer numbers $m \geq \ell \geq 1$ and $x \in \Lambda$ we denote by $R_{x,\ell,m}$ a rectangle on the dual lattice $\Lambda + (1/2, 1/2)$ with side lengths ℓ and m and such that x is the first site of Λ inside the rectangle in lexicographic order. We denote by $\bar{R}_{\ell,m} := \{x \in \Lambda : x \text{ is inside } R_{\ell,m}\}$ the interior of $R_{\ell,m}$. We will drop x from the notation when it will be not necessary to specify the location of the rectangle on the lattice. We say that two rectangles $R_{x,\ell,m}$ and $R_{x',\ell',m'}$ are interacting (resp. non-interacting) iff $d(\bar{R}_{x,\ell,m}, \bar{R}_{x',\ell',m'}) \leq 2$ (resp. $\geq \sqrt{5}$).

We associate a spin variable $\sigma(x) = \pm 1$ to each site $x \in \Lambda$; the space $\{1, -1\}^\Lambda$ of configurations is denoted by \mathcal{S} . If $\sigma \in \mathcal{S}$ and $X \subset \Lambda$ we denote by σ_X a configuration such that $\sigma_X(x) = \sigma(x)$ for any $x \in X$ and $\sigma_X(x)$ is arbitrary for any $x \in \Lambda \setminus X$.

2.2. Definition of the dynamics

Let $\sigma(x) = \pm 1$, for any $x \in \Lambda$, be a spin variable and let

$$H_\Lambda^{(I),h}(\sigma) \equiv H^{(I)}(\sigma) := - \sum_{\langle x,y \rangle} \sigma(x)\sigma(y) - h \sum_{x \in \Lambda} \sigma(x) \quad (2.1)$$

be the Ising nearest neighbors interaction, with the first sum performed over all the nearest neighbor pairs, $\sigma \in \mathcal{S}$ and $h \in \mathbb{R}$.

Let us introduce the discrete time variable $n = 0, 1, \dots$ and denote by σ_n the system configuration at time n . All the spins are updated simultaneously and independently at every unit time; the conditional probability that the spin at site x takes value $a \in \{-1, +1\}$ at time n , given the configuration at time $n-1$, is

$$\begin{aligned} p_x(a|\sigma_{n-1}) &:= \frac{\exp \left\{ -\beta H^{(I)}(a, (\sigma_{n-1})_{\Lambda \setminus \{x\}}) \right\}}{\exp \left\{ -\beta H^{(I)}(a, (\sigma_{n-1})_{\Lambda \setminus \{x\}}) \right\} + \exp \left\{ -\beta H^{(I)}(-a, (\sigma_{n-1})_{\Lambda \setminus \{x\}}) \right\}} \\ &= \frac{1}{1 + \exp \left\{ -2\beta a (S_{\sigma_{n-1}}(x) + h) \right\}} = \frac{1}{2} \left[1 + a \tanh \beta (S_{\sigma_{n-1}}(x) + h) \right] \end{aligned} \quad (2.2)$$

where $\pm a, (\sigma_{n-1})_{\Lambda \setminus \{x\}}$ are the configurations equal to σ_{n-1} on $\Lambda \setminus \{x\}$ and to $\pm a$ on $\{x\}$,

$$S_\sigma(x) := \sum_{y \in \Lambda: d(x,y)=1} \sigma(y)$$

for any $\sigma \in \mathcal{S}$ and $x \in \Lambda$. The normalization condition $p_x(a|\sigma_{n-1}) + p_x(-a|\sigma_{n-1}) = 1$ is trivially satisfied. Thus the time evolution is defined as a Markov chain on \mathcal{S} with non-zero transition probabilities $P_\Lambda(\eta|\sigma)$ given by

$$P_\Lambda(\eta|\sigma) \equiv P_\Lambda(\sigma, \eta) := \prod_{x \in \Lambda} p_x(\eta(x)|\sigma) \quad \forall \sigma, \eta \in \mathcal{S} \quad . \quad (2.3)$$

It is straightforward [D] that the above Probabilistic Cellular Automaton is reversible with respect to the Gibbs measure $\nu_\Lambda(\sigma) := \exp\{-H_\Lambda(\sigma)\}/Z_\Lambda$ with $Z_\Lambda := \sum_{\eta \in \mathcal{S}} \exp\{-H_\Lambda(\eta)\}$ and

$$H_\Lambda^{\beta,h}(\sigma) \equiv H(\sigma) := -\beta h \sum_{x \in \Lambda} \sigma(x) - \sum_{x \in \Lambda} \log \cosh [\beta (S_\sigma(x) + h)] \quad (2.4)$$

In other words the detailed balance condition

$$P_\Lambda(\sigma, \eta) \exp\{-H_\Lambda(\sigma)\} = P_\Lambda(\eta, \sigma) \exp\{-H_\Lambda(\eta)\} \quad (2.5)$$

is satisfied for any $\sigma, \eta \in \mathcal{S}$. The interaction is short range and it is possible to extract the potentials: for any $\sigma \in \mathcal{S}$ we can write

$$\begin{aligned} H(\sigma) - \text{const} = & -J. \sum_{x \in \Lambda} \sigma(x) - J_{\langle \rangle} \sum_{\langle xy \rangle} \sigma(x) \sigma(y) - J_{\langle \langle \rangle \rangle} \sum_{\langle \langle xy \rangle \rangle} \sigma(x) \sigma(y) \\ & - J_{\frown} \sum_{\frown xyz} \sigma(x) \sigma(y) \sigma(z) - J_{\diamond} \sum_{\diamond xywz} \sigma(x) \sigma(y) \sigma(w) \sigma(z) \end{aligned} \quad (2.6)$$

where the five sums are respectively performed over all the sites in Λ , the pairs of next to the nearest neighbors, the pairs of sites at distance 2, the three site clusters composed of two consecutive not parallel pairs of next to the nearest neighbor sites and, finally, over the four site diamond shaped clusters. The even coupling constants are

$$\begin{aligned} J_{\langle \rangle} = 2 J_{\langle \langle \rangle \rangle} &= \frac{1}{8} \log \frac{\cosh \beta(4+h) \cosh \beta(4-h)}{\cosh^2(\beta h)} \stackrel{\beta \rightarrow \infty}{\sim} \beta - \frac{1}{4} \beta h \\ J_{\diamond} &= \frac{1}{16} \log \frac{\cosh \beta(4-h) \cosh^6(\beta h) \cosh \beta(4+h)}{\cosh^4 \beta(2+h) \cosh^4 \beta(2-h)} \stackrel{\beta \rightarrow \infty}{\sim} -\frac{1}{2} \beta + \frac{3}{8} \beta h \end{aligned} \quad (2.7)$$

while the odd ones are

$$\begin{aligned} J_{\cdot} &= \beta h + \frac{1}{4} \log \frac{\cosh^2 \beta(2+h) \cosh \beta(4+h)}{\cosh^2 \beta(2-h) \cosh \beta(4-h)} \stackrel{\beta \rightarrow \infty}{\sim} \frac{5}{2} \beta h \\ J_{\frown} &= \frac{1}{16} \log \frac{\cosh^2 \beta(2-h) \cosh \beta(4+h)}{\cosh^2 \beta(2+h) \cosh \beta(4-h)} \stackrel{\beta \rightarrow \infty}{\sim} -\frac{1}{8} \beta h \end{aligned} \quad (2.8)$$

2.3. The energy and the zero temperature phase diagram

The definition of ground states is not completely trivial in our model, indeed the hamiltonian H_Λ depends on β . The ground states are those configurations on which the Gibbs measure ν_Λ is concentrated when the limit $\beta \rightarrow \infty$ is considered, so they can be defined as the minima of the energy

$$E_\Lambda^h(\sigma) \equiv E(\sigma) := \lim_{\beta \rightarrow \infty} \frac{H_\Lambda(\sigma)}{\beta} = -h \sum_{x \in \Lambda} \sigma(x) - \sum_{x \in \Lambda} |S_\sigma(x) + h| \quad (2.9)$$

uniformly in $\sigma \in \mathcal{S}$. Notice that it is possible to write $H_\Lambda(\sigma) = \beta E_\Lambda(\sigma) + o(\exp\{-\beta c\})$ for some positive constant c depending on σ .

We consider, now, the case $h = 0$: $E_\Lambda(\sigma) = -\sum_{x \in \Lambda} |S_\sigma(x)|$. It is rather clear that there exist four coexisting minima $+\underline{1}, -\underline{1}, C^e, C^o \in \mathcal{S}$:

$$+\underline{1}(x) = +1, \quad -\underline{1}(x) = -1, \quad C^e(x) = (-1)^{x_1+x_2} \quad \text{and} \quad C^o(x) = (-1)^{x_1+x_2+1} \quad (2.10)$$

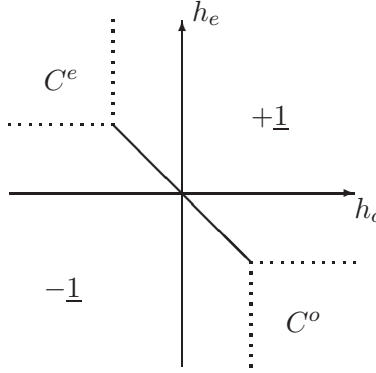


Fig. 2.1: The zero temperature phase diagram in the plane h_o - h_e . The four states $+1$, -1 , C^o and C^e coexist on the solid line whose ending points are $(-4, 4)$ and $(4, -4)$. Each dotted line is the boundary between two regions with different ground states coexisting on the line itself.

for all $x = (x_1, x_2) \in \Lambda$. Notice that C^e and C^o are the chessboard configurations with plus spins respectively on the even and odd sublattices. We define $\mathcal{C} := \{C^o, C^e\}$.

Now, we wonder what happens when $h \neq 0$: a full description of the zero-temperature phase diagram requires the introduction of a staggered magnetic field. We consider the new zero-temperature energy

$$E_{\Lambda}^{h_o, h_e}(\sigma) := - \sum_{x \in \Lambda} h_x \sigma(x) - \sum_{x \in \Lambda} |S_{\sigma}(x) + h_x| \quad , \quad (2.11)$$

where $h_o, h_e \in \mathbb{R}$ and $h_x = h_o$ (resp. $h_x = h_e$) if x belongs to the odd (resp. to the even) sublattice. A simple calculation gives the energy of the four zero-field ground states:

$$\begin{aligned} E_{\Lambda}^{h_o, h_e}(+1) &= -\frac{|\Lambda|}{2} [h_o + h_e + |4 + h_o| + |4 + h_e|] \quad \text{and} \quad E_{\Lambda}^{h_o, h_e}(-1) = E_{\Lambda}^{-h_o, -h_e}(+1) \\ E_{\Lambda}^{h_o, h_e}(C^o) &= -\frac{|\Lambda|}{2} [h_o - h_e + |4 - h_o| + |4 + h_e|] \quad \text{and} \quad E_{\Lambda}^{h_o, h_e}(C^e) = E_{\Lambda}^{-h_o, -h_e}(C^o) \end{aligned} \quad . \quad (2.12)$$

By comparing the four expressions (2.12) one obtains the zero-temperature phase diagram in Fig. 2.1. We note that on the line $h_o = h_e \equiv h$, depending on the sign of the magnetic field the ground state is either $+1$ or -1 ; but at $h = 0$ there are four different coexisting ground states.

2.4. Heuristic description of the low temperature phase diagram

In this section we give a heuristic argument showing that at finite, but very low, temperature the structure of the phase diagram is not changed. More precisely the argument suggests that at $h = 0$ the four states $+1$, -1 , C^o and C^e still coexist [KV, V].

At finite temperature ground states are perturbed because small droplets of different phases show up. The idea is to calculate the energetic cost of a perturbation of one of the four coexisting states via the formation of a square droplet of a different phase. If it results that one of the four ground states is more easily perturbed, then we will conclude that this is the equilibrium phase at finite temperature.

A simple calculation shows that the energy cost of a square droplet of side length n of one of the two homogeneous ground states plunged in one of the two chessboards (or vice versa) is equal to

$8n$. On the other hand if an homogeneous phase is perturbed as above by the other homogeneous phases, or one of the two chessboards is perturbed by the other one, then the energy cost is $16n$.

Hence, from the energetical point of view the most convenient excitations are those in which a homogeneous phase is perturbed by a chessboard or vice versa. Moreover, for each state $-\underline{1}, +\underline{1}, C^e, C^o$ there exist two possible energetically convenient excitations: there is no entropic reason to prefer one of the four ground states to the others when a finite low temperature is considered. This remark strongly suggests that at small finite temperature the four ground states still coexist.

3. Results and heuristics

We pose, now, the question of metastability: let h be positive and small; we prepare the system in the starting configuration $\sigma_0 = -\underline{1}$ and we try to estimate the first time at which the system reaches $+\underline{1}$.

The two chessboard phases coexist at $h = 0$ with the minus and the plus phase: it is natural to wonder if these phases play a role during the escape from the minus metastable phase toward the plus stable phase when the external magnetic field is positive and small.

The main feature of PCA models is that the system can jump from any configuration to any other, in contrast with what happens in serial Glauber dynamics, where transitions are allowed only between configurations differing at most for one spin. We remark that in this model the single spin flip is not a local event, in the sense that its probability depends on all the spin of the lattice. Indeed, given $X \subset \Lambda$ we denote by σ^X the configuration obtained by flipping in σ all the spins at sites $x \in X$; if $X = \{x\}$ for some $x \in \Lambda$, then by abuse of notation we will denote $\sigma^X = \sigma^{\{x\}} = \sigma^x$. Now, by (2.3) we have that

$$P_\Lambda(\sigma, \sigma^X) = \prod_{x \in X} p_x(\sigma^x(x)|\sigma) \prod_{y \in \Lambda \setminus X} p_y(\sigma(y)|\sigma) = \prod_{x \in X} p_x(-\sigma(x)|\sigma) \prod_{y \in \Lambda \setminus X} p_y(\sigma(y)|\sigma) \quad , \quad (3.1)$$

that is the probability to flip the spins inside X depends also on the probability that spins outside X are not flipped. Notice that this is true even if $|X| = 1$, namely if only one spin is flipped.

3.1. Stable configurations

First of all we characterize the stable configurations of the system, namely those configurations $\sigma \in \mathcal{S}$ such that $P_\Lambda(\sigma, \sigma) \rightarrow 1$ in the limit $\beta \rightarrow \infty$. Equivalently, $\sigma \in \mathcal{S}$ is a stable configuration if and only if $P_\Lambda(\sigma, \eta) \rightarrow 0$ in the limit $\beta \rightarrow \infty$ for all $\eta \in \mathcal{S} \setminus \{\sigma\}$.

We discuss, now, the possible single spin events. In Table 1 we consider a site x and we draw all the possible configurations in a five spin cross centered at x . The probability $p_x(+1|\sigma)$ to see $+1$ at x is evaluated (we recall $p_x(-1|\sigma) = 1 - p_x(+1|\sigma)$). From Table 1 it is clear that in the limit $\beta \rightarrow \infty$ the probability associated to a single spin event is either one or zero, in the sequel we will respectively say *high* and *low probability events*. By (3.1) it follows that the same limiting behavior is valid in general for any transition $P_\Lambda(\sigma, \sigma^X)$ with $\sigma \in \mathcal{S}$ and $X \subset \Lambda$; in this sense PCA's are a generalization of deterministic Cellular Automata. We remark that for any $\sigma \in \mathcal{S}$ there exists a unique configuration $\eta \in \mathcal{S}$ such that the transition $\sigma \rightarrow \eta$ happens with high probability, that is $P_\Lambda(\sigma, \eta) \xrightarrow{\beta \rightarrow \infty} 1$. We note, moreover, that $\eta = T\sigma$, where $T : \sigma \in \mathcal{S} \rightarrow T\sigma \in \mathcal{S}$ is the map such that for each $x \in \Lambda$

$$T\sigma(x) := \begin{cases} \sigma^x(x) & \text{if } p_x(\sigma^x(x)|\sigma) \xrightarrow{\beta \rightarrow \infty} 1 \\ \sigma(x) & \text{if } p_x(\sigma^x(x)|\sigma) \xrightarrow{\beta \rightarrow \infty} 0 \end{cases} \quad (3.2)$$

| | | | | | |
|--|-----|--|--|-----|--|
| $\begin{array}{c} - \\ x \\ - \end{array}$ | $-$ | $\frac{1}{1+e^{2\beta(4-h)}} \simeq e^{-2\beta(4-h)}$ | $\begin{array}{c} - \\ x \\ - \end{array}$ | $+$ | $\frac{1}{1+e^{2\beta(2-h)}} \simeq e^{-2\beta(2-h)}$ |
| $\begin{array}{c} - \\ x \\ + \end{array}$ | $+$ | $\frac{1}{1+e^{-2\beta h}} \simeq 1 - e^{-2\beta h}$ | $\begin{array}{c} - \\ x \\ + \end{array}$ | $+$ | $\frac{1}{1+e^{-2\beta(2+h)}} \simeq 1 - e^{-2\beta(2+h)}$ |
| $\begin{array}{c} + \\ x \\ + \end{array}$ | $+$ | $\frac{1}{1+e^{-2\beta(4+h)}} \simeq 1 - e^{-2\beta(4+h)}$ | | | |

Tab. 1: Probabilities for single spin events: probability to see +1 at site x at time t , with the neighboring configuration at time $t-1$ drawn in the picture.

that is at each site we do the right thing in the sense of following the drift. We can say that $\sigma \in \mathcal{S}$ is a stable configuration iff $\sigma = T\sigma$.

In order to characterize the stable states of the model we need few more definitions: let $C \in \mathcal{C} = \{C^o, C^e\}$, we denote by $\mathcal{S}_C \subset \mathcal{S}$ the set of configurations with a well defined sea of chessboard C . Similarly we define $\mathcal{S}_{-1}, \mathcal{S}_{+1} \subset \mathcal{S}$ and we set $\mathcal{S}_C := \mathcal{S}_{C^o} \cup \mathcal{S}_{C^e}$. More precisely, for each $\alpha \in \{-1, +1\}$, for each $\sigma \in \mathcal{S}_\alpha$ there exists a percolating cluster $X \subset \Lambda$ such that $\sigma_X = \alpha_X$ and $\sigma_X = (T^n \sigma)_X$ for all $n \geq 1$; for each $\alpha \in \{C^e, C^o\}$, for each $\sigma \in \mathcal{S}_\alpha$ there exists a percolating cluster $X \subset \Lambda$ such that $\sigma_X = \alpha_X$ and $\sigma_X = (T^{2n} \sigma)_X$ for all $n \geq 1$.

Proposition 3.1 *A configuration $\sigma \in \mathcal{S}_{-1}$ is stable for the PCA (2.3) iff $\sigma(x) = +1$ for all the sites x inside a collection of pairwise non-interacting rectangles of minimal side length $\ell \geq 2$ and $\sigma(x) = -1$ elsewhere. A configuration $\sigma \in \mathcal{S}_{+1}$ is stable iff $\sigma = +1$. There is no stable configuration $\sigma \in \mathcal{S}_C$.*

In other words we can say that the only not trivial stable states are configurations with well separated rectangular droplets of pluses inside the sea of minuses. The Proposition 3.1 follows from [NS1] and Lemma 3.2.

Lemma 3.2 *A configuration $\sigma \in \mathcal{S}$ is stable for the PCA (2.3) iff*

$$p_x(\sigma^x(x)|\sigma) \xrightarrow{\beta \rightarrow \infty} 0 \quad \forall x \in \Lambda$$

Proof of Lemma 3.2. Suppose $p_x(\sigma^x(x)|\sigma) \xrightarrow{\beta \rightarrow \infty} 0$ for all $x \in \Lambda$: let $\eta \in \mathcal{S} \setminus \{\sigma\}$, there exists $X \subset \Lambda$ and $X \neq \emptyset$ such that $\eta = \sigma^X$; thus, by equation (3.1) one has

$$P_\Lambda(\sigma, \eta) = P_\Lambda(\sigma, \sigma^X) = \prod_{x \in X} p_x(\sigma^x(x)|\sigma) \prod_{y \in \Lambda \setminus X} p_y(\sigma(y)|\sigma) \xrightarrow{\beta \rightarrow \infty} 0$$

Suppose σ is a stable configuration: $P(\sigma, \sigma) \rightarrow 1$ in the limit $\beta \rightarrow \infty$, (2.3) and the normalization condition $p_x(\sigma(x)|\sigma) + p_x(\sigma^x(x)|\sigma) = 1$ imply the statement. \square

3.2. Stable pairs and traps

The configurations in which our system can be trapped are not exhausted by the stable configurations. Indeed, let $\sigma \in \mathcal{S}$ and $\eta = T\sigma \neq \sigma$ the unique state reached with high probability starting from σ . If it were $T\eta = \sigma$, then the system would jump back and forth from σ to η with probability going to one in the zero temperature limit; the system would be trapped into a two state loop. Given $\sigma, \eta \in \mathcal{S}$ and $\sigma \neq \eta$, we say that they form a “stable pair” iff $\eta = T\sigma$ and $T\eta = \sigma$. The two chessboard configurations C^o and C^e are a simple example of a stable pair.

We discuss two important properties of the stable pairs. From the detailed balance condition (2.5) it follows that if $\sigma, \eta \in \mathcal{S}$ form a stable pair, then they have the same energy, namely $E_\Lambda(\sigma) = E_\Lambda(\eta)$. Indeed, from (2.5) and (2.9) we have $E_\Lambda(\sigma) - E_\Lambda(\eta) = \lim_{\beta \rightarrow \infty} [H_\Lambda(\sigma) - H_\Lambda(\eta)]/\beta = \lim_{\beta \rightarrow \infty} (1/\beta) \log[P_\Lambda(\sigma, \eta)/P_\Lambda(\eta, \sigma)]$. Now, the fact that σ and η form a stable pair implies $\lim_{\beta \rightarrow \infty} P_\Lambda(\sigma, \eta) = \lim_{\beta \rightarrow \infty} P_\Lambda(\eta, \sigma) = 1$; hence $E_\Lambda(\sigma) = E_\Lambda(\eta)$. By using results in Table 1 one can show that $H_\Lambda(\sigma)$ and $H_\Lambda(\eta)$ differ for a quantity exponentially small in β .

The remark above and the detailed balance condition suggests that the system cannot be trapped in loops longer than two. Indeed, consider a sequence $\sigma_1, \dots, \sigma_n \in \mathcal{S}$ such that $\sigma_{i+1} = T\sigma_i$ for all $i = 1, \dots, n-1$, and suppose, by absurdity, that $\sigma_1 = T\sigma_n$. The property above implies that either $E_\Lambda(\sigma_1) - E_\Lambda(\sigma_n) = c > 0$ or $E_\Lambda(\sigma_1) - E_\Lambda(\sigma_n) = 0$. In the first case from the detailed balance we get $|\log[P_\Lambda(\sigma_1, \sigma_n)/P_\Lambda(\sigma_n, \sigma_1)] - c\beta| \rightarrow 0$ in the limit $\beta \rightarrow \infty$; hence using the hypothesis $\sigma_1 = T\sigma_n$ we easily get an absurd. In the second case the detailed balance implies $|\log[P_\Lambda(\sigma_1, \sigma_n)/P_\Lambda(\sigma_n, \sigma_1)]| \rightarrow 0$, that, together with $\sigma_1 = T\sigma_n$, gives $P_\Lambda(\sigma_1, \sigma_n) \rightarrow 1$, which is absurd because by hypothesis we have $\sigma_2 = T\sigma_1$.

We say that $\sigma \in \mathcal{S}$ is a *trap* if either σ is a stable configuration or the pair $(\sigma, T\sigma)$ is a stable pair. We also let $\mathcal{M} \subset \mathcal{S}$ the collection of all the traps. Now, we give a full description of the stable pairs in $\mathcal{S}_{+1} \cup \mathcal{S}_C \cup \mathcal{S}_{-1}$ (see Fig. 3.2): the most general stable pair living in a sea of minus is made of rectangular flip-flopping droplets of chessboard plunged in the sea of minuses and well separated stable droplets of pluses living inside the sea of minuses or inside a chessboard droplet.

Proposition 3.3 *i) For any $\sigma \in \mathcal{S}_{+1} \setminus \{+1\}$ the pair $(\sigma, T\sigma)$ is not a stable pair. ii) Given $C \in \mathcal{C}$ and $\sigma \in \mathcal{S}_C$ the pair $(\sigma, T\sigma)$ is a stable pair iff there exist $k \geq 0$ pairwise non-interacting rectangles $R_{\ell_1, m_1}, R_{\ell_2, m_2}, \dots, R_{\ell_k, m_k}$, such that $2 \leq \ell_i \leq m_i \leq L-2$ for any $i = 1, \dots, k$, $\sigma_{\mathcal{R}} = +1_{\mathcal{R}}$ (σ coincides with $+1$ inside the rectangles) and $\sigma_{\Lambda \setminus \mathcal{R}} = C_{\Lambda \setminus \mathcal{R}}$ (σ coincides with the chessboard C outside the rectangles), where $\mathcal{R} := \bigcup_{i=1}^k \overline{R}_{\ell_i, m_i}$. iii) Given $\sigma \in \mathcal{S}_{-1}$ the pair $(\sigma, T\sigma)$ is a stable pair iff there exist $k \geq 1$ rectangles $R_{\ell_1, m_1}, R_{\ell_2, m_2}, \dots, R_{\ell_k, m_k}$, with $2 \leq \ell_i \leq m_i \leq L-2$ for any $i = 1, \dots, k$, and there exists an integer $s \in \{1, \dots, k\}$ such that the following conditions are fulfilled:*

1. $\overline{R}_{\ell_i, m_i} \cap \overline{R}_{\ell_j, m_j} = \emptyset$ and $\ell_i \geq 2$ for any $i, j \in \{1, \dots, k\}$;
2. for any $j \in \{1, \dots, s\}$ the family $\{R_{\ell_j, m_j}, R_{\ell_{s+1}, m_{s+1}}, \dots, R_{\ell_k, m_k}\}$ is a family of pairwise non-interacting rectangles;
3. $\sigma_{\Lambda \setminus \mathcal{R}} = -1_{\Lambda \setminus \mathcal{R}}$ where $\mathcal{R} := \bigcup_{i=1}^k \overline{R}_{\ell_i, m_i}$ (σ coincides with -1 outside the rectangles);
4. $\sigma_{\overline{R}_{\ell_j, m_j}} = +1_{\overline{R}_{\ell_j, m_j}}$ for any $j \in \{s+1, \dots, k\}$ (σ is plus inside $R_{\ell_{s+1}, m_{s+1}}, \dots, R_{\ell_k, m_k}$);
5. for any $j \in \{1, \dots, s\}$ there exist $k' \equiv k'(j) \geq 0$ rectangles $R'_{\ell'_1, m'_1} = R'_{\ell'_1, m'_1}(j), \dots, R'_{\ell'_{k'}, m'_{k'}} = R'_{\ell'_{k'}, m'_{k'}}(j)$ such that the following conditions are fulfilled:
 - 5.1. $\overline{R}'_{\ell'_i, m'_i} \subset \overline{R}_{\ell_j, m_j}$ for any $i \in \{1, \dots, k'\}$;
 - 5.2. for any $j = 1, \dots, s$ the family $\{R'_{\ell'_i, m'_i} : i = 1, \dots, k'\}$ (recall $R'_{\ell'_i, m'_i} = R'_{\ell'_i, m'_i}(j)$ for any $i = 1, \dots, k' = k'(j)$) is a family of pairwise non-interacting rectangles;

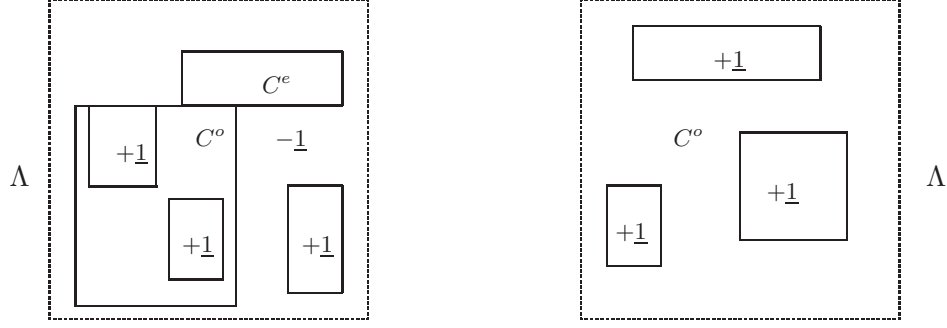


Fig. 3.2: On the left (resp. right) the most general $\sigma \in \mathcal{S}_{-1}$ (resp. \mathcal{S}_C) such that $(\sigma, T\sigma)$ is a stable pair.

5.3. $\sigma_{\mathcal{R}'} = +1_{\mathcal{R}'}$ where $\mathcal{R}' \equiv \mathcal{R}'(j) := \bigcup_{i=1}^{k'} \overline{R'}_{\ell'_i, m'_i}$

5.4. either $\sigma_{\overline{R}_{\ell_j, m_j} \setminus \mathcal{R}'} = C_{\overline{R}_{\ell_j, m_j} \setminus \mathcal{R}'}^o$ or $\sigma_{\overline{R}_{\ell_j, m_j} \setminus \mathcal{R}'} = C_{\overline{R}_{\ell_j, m_j} \setminus \mathcal{R}'}^e$;

6. for any $i, j \in \{1, \dots, s\}$ the two rectangles R_{ℓ_j, m_j} and R_{ℓ_i, m_i} must be non-interacting if $\sigma_{\overline{R}_{\ell_j, m_j} \setminus \mathcal{R}'(j)} = \sigma_{\overline{R}_{\ell_i, m_i} \setminus \mathcal{R}'(i)}$.

3.3. Basic tools

In this Section we discuss the main tools that will be used in the following: first of all we notice that in our model the difference of energy between two configurations $\sigma, \eta \in \mathcal{S}$ is not sufficient to say if the system prefers to jump from σ to η or vice versa. Indeed, there exist pairs of configurations $\sigma, \eta \in \mathcal{S}$ such that the system sees a sort of energetic barrier both in the $\sigma \rightarrow \eta$ and in the $\eta \rightarrow \sigma$ transition. Let us define a sort of “communicating height” $H(\sigma, \eta)$ for each pair $(\sigma, \eta) \in \mathcal{S} \times \mathcal{S}$ of the configuration space such that

$$P_{\Lambda}(\sigma, \eta) =: e^{-[H(\sigma, \eta) - H(\sigma)]} \quad . \quad (3.3)$$

More precisely, we consider a new hamiltonian $H : \mathcal{S} \times \mathcal{S} \cup \mathcal{S} \rightarrow \mathbb{R}$ defined as in (2.4) for any $\sigma \in \mathcal{S}$ and such that

$$H(\sigma, \eta) := H(\sigma) - \log P_{\Lambda}(\sigma, \eta) \quad . \quad (3.4)$$

Note that, by virtue of the detailed balance principle (2.5), we have $H(\sigma, \eta) = H(\eta, \sigma)$. Remark: if either $P_{\Lambda}(\sigma, \eta)$ or $P_{\Lambda}(\eta, \sigma)$ tends to zero in the limit $\beta \rightarrow \infty$ then $H(\sigma, \eta) = \max\{H(\sigma), H(\eta)\} + o(\exp\{-\beta c\})$, for some strictly positive constant c ; in other words in these cases the energetic barrier seen by the system is exactly the difference of energy between the two configurations.

We notice that in [OS] it has already been remarked that the communicating heights allow to define the most general kind of reversible dynamics (see Section 3 in [OS]). Now we want to restate in this setup some of the results of [OS] that will be our basic tools in next sections.

We say that a configuration $\sigma \in \mathcal{S}$ is a local minimum of the energy iff $H(\sigma, \eta) - H(\sigma) > 0$ for any $\eta \in \mathcal{S} \setminus \{\sigma\}$. The local minima of the energy are nothing but the stable configurations defined above. A sequence of configurations $\omega = \{\omega_0, \dots, \omega_n\}$ is called a “path”; $|\omega|$ is the number of configurations in the path. We call “height along the path ω ” the real number

$$\Phi_{\omega} := \max_{i=1, \dots, |\omega|} H(\omega_{i-1}, \omega_i) \quad . \quad (3.5)$$

Given two configurations $\sigma, \eta \in \mathcal{S}$ we denote by $\Theta(\sigma, \eta)$ the set of all the paths $\omega = \{\omega_0, \dots, \omega_n\}$ such that $\omega_0 = \sigma$ and $\omega_n = \eta$. The “minimal height” (minmax) between σ and η is defined as

$$\Phi(\sigma, \eta) := \min_{\omega \in \Theta(\sigma, \eta)} \Phi_\omega = \min_{\omega \in \Theta(\sigma, \eta)} \max_{i=1, \dots, |\omega|} H(\omega_{i-1}, \omega_i) \quad . \quad (3.6)$$

We remark that the function $\Phi : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$ is symmetric, namely $\Phi(\sigma, \eta) = \Phi(\eta, \sigma)$ for any $\sigma, \eta \in \mathcal{S}$.

We give, now, the important notion of cycle: we say that $\mathcal{A} \subset \mathcal{S}$ is a cycle iff for each $\sigma, \eta \in \mathcal{A}$

$$\Phi(\sigma, \eta) < \min_{\zeta \in \mathcal{S} \setminus \mathcal{A}} \Phi(\sigma, \zeta) \quad . \quad (3.7)$$

In other words starting from any configuration in the cycle \mathcal{A} , the energetic barrier that must be bypassed to visit any other configuration in \mathcal{A} is smaller than the one seen to exit the cycle itself.

Given a cycle $\mathcal{A} \subset \mathcal{S}$ we denote by $F(\mathcal{A})$ the set of the minima of the energy in \mathcal{A} , namely

$$F(\mathcal{A}) := \{\sigma \in \mathcal{A} : \min_{\eta \in \mathcal{A}} H(\eta) = H(\sigma)\} \quad ; \quad (3.8)$$

we also write $H(F(\mathcal{A})) = H(\eta)$ with $\eta \in F(\mathcal{A})$. Given $\eta \in F(\mathcal{A})$ we define

$$\Phi(\mathcal{A}) := \min_{\zeta \in \mathcal{S} \setminus \mathcal{A}} \Phi(\eta, \zeta) \quad (3.9)$$

(it is trivial that $\Phi(\mathcal{A})$ does not depend on the choice of $\eta \in F(\mathcal{A})$), and the set

$$U(\mathcal{A}) := \{\zeta \in \mathcal{S} \setminus \mathcal{A} : \exists \eta \in \mathcal{A} \text{ such that } H(\eta, \zeta) = \Phi(\mathcal{A})\} \quad . \quad (3.10)$$

Now, for any $\eta \in \mathcal{S}$ let \mathbb{P}_η be the probability over the process when the system is prepared in $\sigma_0 = \eta$ and

$$\tau_{\mathcal{D}} := \inf\{n \geq 0 : \sigma_n \in \mathcal{D}\} \quad (3.11)$$

for any $\mathcal{D} \subset \mathcal{S}$. We restate, without proof, some of the results of [OS] that we will use in the sequel:

Lemma 3.4 *Given $\mathcal{G} \subset \mathcal{S}$, let $\sigma \in \mathcal{G}$ and $\sigma' \in \mathcal{S} \setminus \mathcal{G}$ such that: i) there exist $\sigma^* \in \mathcal{S} \setminus \mathcal{G}$ and a path $\omega = \{\omega_0 = \sigma, \dots, \omega_n = \sigma^*\}$ such that $\omega_i \in \mathcal{G}$ and $H(\omega_{i-1}, \omega_i) < H(\omega_{n-1}, \omega_n) =: \Gamma$ for any $i = 1, \dots, n-1$; ii) there exists a path $\omega' = \{\omega'_0 = \sigma^*, \dots, \omega'_n = \sigma'\}$ such that $\omega'_i \in \mathcal{S} \setminus \mathcal{G}$ and $H(\omega'_{i-1}, \omega'_i) < \Gamma$ for any $i = 1, \dots, n$; iii) $\min_{\sigma \in \mathcal{G}, \eta \in \mathcal{S} \setminus \mathcal{G}} H(\sigma, \eta) \geq \Gamma$ if and only if $\sigma = \omega_{n-1}$ and $\eta = \omega_n$. If we define*

$$\mathcal{A} := \{\eta \in \mathcal{S} : \exists \omega = \{\omega_0 = \eta, \dots, \omega_n = \sigma\} \text{ such that } \omega_1, \dots, \omega_{n-1} \in \mathcal{G} \text{ and } \Phi_\omega < \Gamma\}$$

then i) $\mathcal{A} \subset \mathcal{G}$; ii) \mathcal{A} is a cycle with $\Phi(\mathcal{A}) = \Gamma$ and $\sigma^ \in U(\mathcal{A})$; iii) $\Phi(\sigma, \sigma') = \Gamma$ (that is Γ is the minmax between σ and σ').*

Lemma 3.5 *Given a cycle $\mathcal{A} \subset \mathcal{S}$,*

i) for all $\varepsilon > 0$ and for all $\sigma \in F(\mathcal{A})$

$$\mathbb{P}_\sigma \left(\exp\{\Phi(\mathcal{A}) - H(F(\mathcal{A})) - \beta\varepsilon\} < \tau_{\mathcal{S} \setminus \mathcal{A}} < \exp\{\Phi(\mathcal{A}) - H(F(\mathcal{A})) + \beta\varepsilon\} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (3.12)$$

ii) there exists $\delta > 0$ such that for any $\sigma, \eta \in \mathcal{A}$

$$\mathbb{P}_\sigma \left(\tau_\eta < \tau_{\mathcal{S} \setminus \mathcal{A}}, \tau_\eta < \exp\{\Phi(\mathcal{A}) - H(F(\mathcal{A})) - \beta\delta\} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (3.13)$$

iii) for any $\sigma \in \mathcal{A}$

$$\mathbb{P}_\sigma \left(\sigma_{\tau_{\mathcal{S} \setminus \mathcal{A}}} \in U(\mathcal{A}) \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (3.14)$$

iv) for any $\sigma \in \mathcal{A}$, $\eta \in U(\mathcal{A})$, $\varepsilon > 0$ and β sufficiently large

$$\mathbb{P}_\sigma \left(\sigma_{\tau_{\mathcal{S} \setminus \mathcal{A}}} = \eta \right) \geq e^{-\beta\varepsilon} \quad (3.15)$$

We note that the lower bound on $\tau_{\mathcal{S} \setminus \mathcal{A}}$ in the statement i) in Lemma 3.5 is an easy consequence of the reversibility property:

Lemma 3.6 For any $\sigma, \eta \in \mathcal{S}$ such that $\Phi(\sigma, \eta) - H(\sigma) > 0$ and for any $\delta > 0$

$$\mathbb{P}_\sigma (\tau_\eta > \exp\{\Phi(\sigma, \eta) - H(\sigma) - \beta\delta\}) \xrightarrow{\beta \rightarrow \infty} 1 \quad . \quad (3.16)$$

Few important remarks which are very peculiar of our PCA model. As it has been noticed above if the system is in the state $\eta \in \mathcal{S}$, then there exists a unique configuration where it jumps with high probability. This configuration has been denoted by $T\eta$. Thus, given $\eta \in \mathcal{S}$ we define the *downhill* path starting from η as the unique path $\omega = \{\omega_0, \dots, \omega_n\}$ such that $\omega_0 = \eta$, $T\omega_{i-1} = \omega_i$ for any $i = 1, \dots, n$, and ω_n is a trap; we also set $\hat{\eta} := \omega_n$. We remark that to each $\eta \in \mathcal{S}$ we can associate either a unique stable configuration or a unique stable pair. We define the *basin of attraction* of a trap $\eta \in \mathcal{M}$ as the set

$$\mathcal{B}(\eta) := \{\zeta \in \mathcal{S} : \hat{\zeta} = \eta\} \quad (3.17)$$

and the *truncated* basin of attraction $\bar{\mathcal{B}}(\sigma) \subset \mathcal{B}(\sigma)$ as the set of all the configurations $\eta \in \mathcal{B}(\sigma)$ such that

$$\Phi(\eta, \sigma) < \min_{\zeta \in \mathcal{S} \setminus \mathcal{B}(\eta)} \Phi(\eta, \zeta) \quad . \quad (3.18)$$

It can be easily proven that $\bar{\mathcal{B}}(\eta)$ is a cycle.

In the following we will often have to evaluate $\Upsilon(\eta) := \min_{\zeta \in \mathcal{S} \setminus \mathcal{B}(\eta)} \Phi(\eta, \zeta)$ for some trap $\eta \in \mathcal{M}$. A convenient way to proceed is the following: say that a path $\omega = \{\omega_0, \dots, \omega_n\}$ is uphill iff the path $\omega' = \{\omega'_0, \dots, \omega'_n\}$, where $\omega'_i = \omega_{n-i}$ for any $i = 0, \dots, n$, is downhill. Consider the set $\Xi(\eta)$ of paths $\omega = \{\omega_0, \dots, \omega_n\}$ such that $\omega_0 = \eta$, $\{\omega_0, \dots, \omega_{n-1}\}$ is an uphill path in $\mathcal{B}(\eta)$ and $\omega_n \in \mathcal{S} \setminus \mathcal{B}(\eta)$. We remark that $\Upsilon(\eta)$, in words the barrier that must be bypassed to exit from the basin $\mathcal{B}(\eta)$, is given by

$$\Upsilon(\eta) = \min_{\omega \in \Xi(\eta)} \Phi_\omega \quad . \quad (3.19)$$

3.4. Behavior of traps

In this subsection we clarify the geometrical conditions for the shrinking or the growing of a trap, that is we study the evolution of the system prepared in a stable configuration or in a stable pair. We let $\lambda := [2/h] + 1$.

We first consider the case of a single rectangular droplet of chessboard or pluses in the sea of minuses; we show that if the droplet is small enough, namely its shortest side is smaller than λ , then it tends to shrink, otherwise it tends to grow.

Proposition 3.7 Let $\zeta \in \{+1, C^e, C^o\}$; $\eta \in \mathcal{M}$ such that there exists a rectangle $R_{\ell, m}$, with $2 \leq \ell \leq m$, such that $\eta_{\Lambda \setminus \bar{R}_{\ell, m}} = -1_{\Lambda \setminus \bar{R}_{\ell, m}}$ and $\eta_{\bar{R}_{\ell, m}} = \zeta_{\bar{R}_{\ell, m}}$. Thus

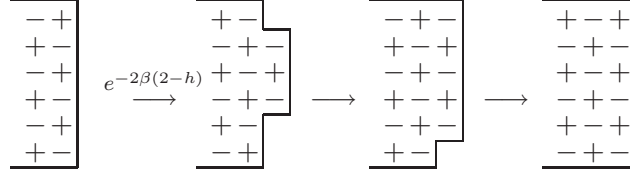


Fig. 3.3: Growth of a chessboard droplet inside the sea of minuses: appearing of a protuberance.

i) if $\ell < \lambda$, then η is subcritical, that is $\mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_{\mathcal{S}_C \cup \mathcal{S}_{+\underline{1}}}) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta(\exp\{2\beta h(\ell - 1) - \beta\varepsilon\} < \tau_{-\underline{1}} < \exp\{2\beta h(\ell - 1) + \beta\varepsilon\}) \xrightarrow{\beta \rightarrow \infty} 1 \quad ; \quad (3.20)$$

ii) if $\ell \geq \lambda$, then η is supercritical, that is $\mathbb{P}_\eta(\tau_{\mathcal{S}_C \cup \mathcal{S}_{+\underline{1}}} < \tau_{-\underline{1}}) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta(\exp\{2\beta(2 - h) - \beta\varepsilon\} < \tau_{\mathcal{S}_C \cup \mathcal{S}_{+\underline{1}}} < \exp\{2\beta(2 - h) + \beta\varepsilon\}) \xrightarrow{\beta \rightarrow \infty} 1 \quad . \quad (3.21)$$

Similar results can be stated in the case of a single droplet trap plunged inside the sea of chessboard.

Proposition 3.8 *Let $C \in \mathcal{C}$ and $\eta \in \mathcal{M}$ a trap such that there exists a rectangle $R_{\ell,m}$, with $2 \leq \ell \leq m$, and $\eta_{\Lambda \setminus \overline{R}_{\ell,m}} = C_{\Lambda \setminus \overline{R}_{\ell,m}}$ and $\eta_{\overline{R}_{\ell,m}} = +\underline{1}_{\overline{R}_{\ell,m}}$. Thus*

i) if $\ell < \lambda$, then $\mathbb{P}_\eta(\tau_C < \tau_{+\underline{1}}) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta(\exp\{2\beta h(\ell - 1) - \beta\varepsilon\} < \tau_C < \exp\{2\beta h(\ell - 1) + \beta\varepsilon\}) \xrightarrow{\beta \rightarrow \infty} 1 \quad ; \quad (3.22)$$

ii) if $\ell \geq \lambda$, then $\mathbb{P}_\eta(\tau_{+\underline{1}} < \tau_C) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta(\exp\{2\beta(2 - h) - \beta\varepsilon\} < \tau_{+\underline{1}} < \exp\{2\beta(2 - h) + \beta\varepsilon\}) \xrightarrow{\beta \rightarrow \infty} 1 \quad . \quad (3.23)$$

Now, we give two heuristic arguments supporting the Propositions above. We consider the case $\ell = m$ even and $\eta_{\overline{R}_{\ell,\ell}} = C_{\overline{R}_{\ell,\ell}}^o$: by using (2.9) one can show that the energy of η , with respect to the configuration $-\underline{1}$, is

$$E_\Lambda^h(\eta) - E_\Lambda^h(-\underline{1}) = -2h\ell^2 + 8\ell \quad . \quad (3.24)$$

Thus, $E_\Lambda^h(\eta) - E_\Lambda^h(-\underline{1})$ is a parabola whose maximum is achieved at $\ell = 2/h$ suggesting the conjecture that the critical length is $2/h$.

A dynamical argument strengthens this conjecture. Consider the most efficient growing mechanism: from results in Table 1 this mechanism is the appearance of a single plus protuberance adjacent to one of the four sides of the rectangle. The probability associated to such an event is $\exp\{-2\beta(2 - h)\}$, so that the typical time to see this event is $\tau_{\text{gr}} \sim \exp\{2\beta(2 - h)\}$. In Fig. 3.3 it is shown that once the protuberance has appeared on one of the four sides of the rectangle, with high probability a new slice is filled with chessboard.

Now we have to look for the efficient shrinking mechanism: Table 1 suggests this mechanism is the “minus corner persistence”, that is a minus spin on one of the four corners of the rectangle is kept fixed during the flip-flop of the stable pair. By reiterating this mechanism $\ell - 1$ times (see

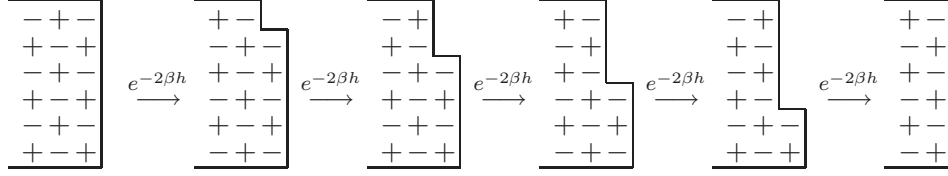


Fig. 3.4: Shrinking of a chessboard droplet inside the sea of minuses: persistence of a minus corner.

Fig. 3.4) a full slice of the droplet is erased. In terms of probability each step costs $\exp\{-2\beta h\}$, hence the typical shrinking time is $\tau_{\text{sh}} \sim \exp\{2\beta h(\ell - 1)\}$. By comparing τ_{gr} and τ_{sh} we find that the critical length should be $2/h$.

A similar argument can be done in the case of a plus droplet: the growing mechanism is still the formation of a protuberance. About the shrinking mechanism: after a first step of “corner erosion”, like in the standard Glauber case a corner is flipped into minus, one minus spin appears at the corner. The best thing to do, as a second step, is to flip simultaneously both the minus spin and its adjacent plus spin. This event costs still $\exp\{-2\beta h\}$ and results in a shift of the minus “lacuna” on the side of the rectangle. By iterating this mechanisms a sort of merlon is formed and a stable pair is reached in a typical time $\tau_{\text{sh}} \sim \exp\{2\beta h(\ell - 1)\}$.

A stronger version of the above Propositions can be proved; it is possible to describe in detail the way in which droplets shrink or grow. Indeed we state:

Proposition 3.9 *i) Let $C \in \mathcal{C}$, $\eta \in \mathcal{M}$ such that there exists a rectangle $R_{\ell,m}$, with $\ell \leq m$, such that $\eta_{\Lambda \setminus \overline{R}_{\ell,m}} = -\underline{1}_{\Lambda \setminus \overline{R}_{\ell,m}}$ and $\eta_{\overline{R}_{\ell,m}} = C_{\overline{R}_{\ell,m}}$. Let \mathcal{A}' the set of traps $\sigma \in \mathcal{M}$ such that there exists a rectangle R with side lengths $(\ell, m+1)$ or $(\ell+1, m)$ such that $\overline{R} \supset \overline{R}_{\ell,m}$, $\sigma_{\Lambda \setminus \overline{R}} = -\underline{1}_{\Lambda \setminus \overline{R}}$ and $\sigma_{\overline{R}} = C_{\overline{R}}$. Let \mathcal{A}'' the set of traps $\sigma \in \mathcal{M}$ such that there exists a rectangle R with side lengths $(\ell, m-1)$ such that $\overline{R} \subset \overline{R}_{\ell,m}$, $\sigma_{\Lambda \setminus \overline{R}} = -\underline{1}_{\Lambda \setminus \overline{R}}$ and $\sigma_{\overline{R}} = C_{\overline{R}}$. If $\lambda \leq \ell$ then*

$$\Upsilon(\eta) = H(\eta) + 2\beta(2 - h) \quad \text{and} \quad \mathbb{P}_\eta \left(\sigma_{\tau_{S \setminus \mathcal{B}(\eta)}} \in \bigcup_{\sigma \in \mathcal{A}'} \mathcal{B}(\sigma) \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad ,$$

that is starting from η the system exits $\mathcal{B}(\eta)$ and enters into one of the basins $\mathcal{B}(\sigma)$, for some $\sigma \in \mathcal{A}'$. If $\ell < \lambda$ then

$$\Upsilon(\eta) = H(\eta) + 2\beta h(\ell - 1) \quad \text{and} \quad \mathbb{P}_\eta \left(\sigma_{\tau_{S \setminus \mathcal{B}(\eta)}} \in \mathcal{A}'' \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad ,$$

that is starting from η the system exits $\mathcal{B}(\eta)$ and reaches directly one of the traps σ in \mathcal{A}'' .

ii) Let $C \in \mathcal{C}$, $\eta \in \mathcal{M}$ such that there exists a rectangle $R_{\ell,m}$, with $\ell \leq m$, such that $\eta_{\Lambda \setminus \overline{R}_{\ell,m}} = C_{\Lambda \setminus \overline{R}_{\ell,m}}$ and $\eta_{\overline{R}_{\ell,m}} = +\underline{1}_{\overline{R}_{\ell,m}}$. Let \mathcal{A}' the set of traps $\sigma \in \mathcal{M}$ such that there exists a rectangle R with side lengths $(\ell, m+1)$ or $(\ell+1, m)$ such that $\overline{R} \supset \overline{R}_{\ell,m}$, $\sigma_{\Lambda \setminus \overline{R}} = C_{\Lambda \setminus \overline{R}}$ and $\sigma_{\overline{R}} = +\underline{1}_{\overline{R}}$. Let \mathcal{A}'' the set of traps $\sigma \in \mathcal{M}$ such that there exists a rectangle R with side lengths $(\ell, m-1)$ such that $\overline{R} \subset \overline{R}_{\ell,m}$, $\sigma_{\Lambda \setminus \overline{R}} = C_{\Lambda \setminus \overline{R}}$ and $\sigma_{\overline{R}} = +\underline{1}_{\overline{R}}$. If $\lambda \leq \ell$ then

$$\Upsilon(\eta) = H(\eta) + 2\beta(2 - h) \quad \text{and} \quad \mathbb{P}_\eta \left(\sigma_{\tau_{S \setminus \mathcal{B}(\eta)}} \in \bigcup_{\sigma \in \mathcal{A}'} \mathcal{B}(\sigma) \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad ,$$

that is starting from η the system exits $\mathcal{B}(\eta)$ and enters into one of the basins $\mathcal{B}(\sigma)$, for some $\sigma \in \mathcal{A}'$. If $\ell < \lambda$ then

$$\Upsilon(\eta) = H(\eta) + 2\beta h(\ell - 1) \quad \text{and} \quad \mathbb{P}_\eta \left(\sigma_{\tau_{\mathcal{S} \setminus \mathcal{B}(\eta)}} \in \mathcal{A}'' \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad ,$$

that is starting from η the system exits $\mathcal{B}(\eta)$ and reaches directly one of the traps σ in \mathcal{A}'' .

Finally, we state under which conditions a general trap shrinks. A trap is made of rectangles of pluses or chessboard inside a minus or chessboard sea. The idea is that the configuration shrinks iff each single rectangular cluster shrinks. Note that the following proposition strictly contains Propositions 3.7 and 3.8.

Proposition 3.10 *i) Let $C \in \mathcal{C}$ and $\eta \in \mathcal{M}_C$. There exist $k \geq 1$ pairwise non-interacting rectangles $R_{\ell_1, m_1}, \dots, R_{\ell_k, m_k}$ such that $2 \leq \ell_i \leq m_i \leq L - 2$ for any $i = 1, \dots, k$, $\eta_{\mathcal{R}} = +\underline{1}_{\mathcal{R}}$ and $\eta_{\Lambda \setminus \mathcal{R}} = C_{\Lambda \setminus \mathcal{R}}$ where $\mathcal{R} := \bigcup_{i=1}^k \overline{R}_{\ell_i, m_i}$. Thus, if $\ell_i < \lambda$ for any $i = 1, \dots, k$, then $\mathbb{P}_\eta(\tau_C < \tau_{+\underline{1}}) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$*

$$\mathbb{P}_\eta \left(\exp \{2\beta h(\ell - 1) - \beta\varepsilon\} < \tau_C < \exp \{2\beta h(\ell - 1) + \beta\varepsilon\} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad , \quad (3.25)$$

where $\ell := \max\{\ell_1, \dots, \ell_k\}$. If there exists $j \in \{1, \dots, k\}$ such that $\ell_j \geq \lambda$, then $\mathbb{P}_\eta(\tau_{+\underline{1}} < \tau_C) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta \left(\exp \{2\beta(2 - h) - \beta\varepsilon\} < \tau_{+\underline{1}} < \exp \{2\beta(2 - h) + \beta\varepsilon\} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (3.26)$$

(note that in the case $k = 1$ we recover Proposition 3.8). ii) We consider, now, a situation where in a sea of minuses there are rectangles of chessboard and, possibly, rectangles of pluses inside the sea of minuses or inside the chessboard droplets. More precisely, let $\eta \in \mathcal{M}$. We suppose that there exist $k \geq 1$ rectangles $R_{\ell_1, m_1}, \dots, R_{\ell_k, m_k}$, with $2 \leq \ell_i \leq m_i \leq L - 2$ for any $i = 1, \dots, k$, and there exists an integer $s \in \{1, \dots, k\}$ such that the conditions of point (iii) in Proposition 3.3 are satisfied (note that in the case $s = 0$ the trap is a stable configuration; in the case $s = 0$ and $k = 1$ Proposition 3.7 is recovered). Thus, if $\ell_i < \lambda$ for any $i = 1, \dots, k$, then $\mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_{\mathcal{S}_C \cup \mathcal{S}_{+\underline{1}}}) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta \left(\exp \{2\beta h(\ell - 1) - \beta\varepsilon\} < \tau_{-\underline{1}} < \exp \{2\beta h(\ell - 1) + \beta\varepsilon\} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad , \quad (3.27)$$

where $\ell := \max\{\ell_1, \dots, \ell_k\}$. If there exists $j \in \{1, \dots, k\}$ such that $\ell_j \geq \lambda$, then η is supercritical, that is $\mathbb{P}_\eta(\tau_{\mathcal{S} \setminus \mathcal{S}_{-\underline{1}}} < \tau_{-\underline{1}}) \xrightarrow{\beta \rightarrow \infty} 1$, and for any $\varepsilon > 0$

$$\mathbb{P}_\eta \left(\exp \{2\beta(2 - h) - \beta\varepsilon\} < \tau_{\mathcal{S} \setminus \mathcal{S}_{-\underline{1}}} < \exp \{2\beta(2 - h) + \beta\varepsilon\} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad . \quad (3.28)$$

3.5. Exit from the metastable phase

We can now give the theorem describing the exit from the metastable state. Suppose that the system is prepared in the metastable state, $\sigma_0 = -\underline{1}$, in the following theorem we state that the first exit time $\tau_{+\underline{1}}$ is exponentially large in β and we find its order of magnitude. Moreover, we state that before reaching $+\underline{1}$ the system visits $\mathcal{C} = \{C^o, C^e\}$ and that the typical time to jump from $-\underline{1}$ to the chessboards is the same as the time needed to jump from the chessboards to $+\underline{1}$. More precisely, let us denote by $\mathcal{Q}_{-\underline{1}} \subset \mathcal{S}$ the set of configurations $\eta \in \mathcal{S}$ such that there exists a

rectangle $R_{\lambda,\lambda}$ such that $\eta_{\Lambda \setminus \overline{R}_{\lambda,\lambda}} = -\underline{1}_{\Lambda \setminus \overline{R}_{\lambda,\lambda}}$ and $\eta_{\overline{R}_{\lambda,\lambda}} = C_{\overline{R}_{\lambda,\lambda}}$ with $C \in \mathcal{C}$. Let $\eta \in \mathcal{Q}_{-\underline{1}}$, we call “protocritical height” the energy

$$\Gamma := E_{\Lambda}^h(\eta) - E_{\Lambda}^h(-\underline{1}) + 2h(\lambda - 1) = -2h\lambda^2 + 2\lambda(4 + h) - 2h \quad , \quad (3.29)$$

where the second equality follows from (3.24). In some sense $\beta\Gamma$ is the communication height between the largest subcritical droplet and the smallest supercritical droplet; more precisely: let $\mathcal{O}_{-\underline{1}} \subset \mathcal{S}$ the set of configurations $\eta \in \mathcal{S}_{-\underline{1}}$ such that there exists a rectangle $R_{\lambda-1,\lambda}$ such that $\eta_{\Lambda \setminus \overline{R}_{\lambda-1,\lambda}} = -\underline{1}_{\Lambda \setminus \overline{R}_{\lambda-1,\lambda}}$ and $\eta_{\overline{R}_{\lambda-1,\lambda}} = C_{\overline{R}_{\lambda-1,\lambda}}$ with $C \in \mathcal{C}$.

Let $\eta \in \mathcal{O}_{-\underline{1}}$: we call *protocritical* droplet corresponding to η one of the configurations obtained by flipping in η a minus spin external to $R_{\lambda-1,\lambda}$ and adjacent to one of the plus spins of the internal chessboard and all the spins associated to sites inside $R_{\lambda-1,\lambda}$. We let $\pi_{-\underline{1}}(\eta)$ the set of protocritical droplets corresponding to η and $\mathcal{P}_{-\underline{1}} := \cup_{\eta \in \mathcal{O}_{-\underline{1}}} \pi_{-\underline{1}}(\eta)$, the collection of the protocritical droplets. It is easy to check that $H(\eta, \zeta) = \Gamma$ for any $\eta \in \mathcal{O}_{-\underline{1}}$ and $\zeta \in \pi_{-\underline{1}}(\eta)$.

Theorem 3.11 *With the notation introduced above: i) the system visits $\mathcal{P}_{-\underline{1}}$ before visiting \mathcal{C} , namely*

$$\mathbb{P}_{-\underline{1}}(\tau_{\mathcal{P}_{-\underline{1}}} < \tau_{\mathcal{C}}) \xrightarrow{\beta \rightarrow \infty} 1 \quad ;$$

ii) the system visits \mathcal{C} before visiting $+\underline{1}$, namely

$$\mathbb{P}_{-\underline{1}}(\tau_{\mathcal{C}} < \tau_{+\underline{1}}) \xrightarrow{\beta \rightarrow \infty} 1 \quad ;$$

iii) for any $\varepsilon > 0$

$$\mathbb{P}_{-\underline{1}}(e^{\beta\Gamma - \beta\varepsilon} < \tau_{\mathcal{C}} < e^{\beta\Gamma + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1 \quad ;$$

iv) for any $\varepsilon > 0$

$$\mathbb{P}_{-\underline{1}}(e^{\beta\Gamma - \beta\varepsilon} < \tau_{+\underline{1}} < e^{\beta\Gamma + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1 \quad .$$

The proof of Theorem 3.11 will be the argument of Section 4.

In the above theorem we have stated that during the exit from the metastable $-\underline{1}$ state, the system visits the competing metastable state \mathcal{C} and, finally, reaches the stable state $+\underline{1}$. Now, we want to give a more precise description of the path followed by the system during its exit from $-\underline{1}$; first of all we define a suitable tube $\mathcal{T}_{-\underline{1}}$. Let $\mathcal{O}_{-\underline{1}}^{(0)} := \mathcal{O}_{-\underline{1}}$ the set of $\lambda \times (\lambda - 1)$ chessboard droplets in the sea of minuses. For each $k = 0, 1, \dots, 2\lambda - 6$ we define recursively the sets $\mathcal{O}_{-\underline{1}}^{(k)}(\eta^{(k-1)})$, where $\eta^{(k-1)} \in \mathcal{O}_{-\underline{1}}^{(k-1)}(\eta^{(k-2)})$: let $\eta^{(k-1)} \in \mathcal{O}_{-\underline{1}}^{(k-1)}(\eta^{(k-2)})$ a configuration such that there exists a rectangle $R_{\ell,m}$, with $2 \leq \ell \leq m$, such that $\eta_{\overline{R}_{\ell,m}}^{(k-1)} = C_{\overline{R}_{\ell,m}}$, with $C \in \mathcal{C}$, and $\eta_{\Lambda \setminus \overline{R}_{\ell,m}}^{(k-1)} = -\underline{1}_{\Lambda \setminus \overline{R}_{\ell,m}}$. Then we define $\mathcal{O}_{-\underline{1}}^{(k)}(\eta^{(k-1)})$ as the collection of configurations ζ such that there exists a rectangle $R_{\ell,m-1}$ such that $\overline{R}_{\ell,m-1} \subset \overline{R}_{\ell,m}$, $\zeta_{\Lambda \setminus \overline{R}_{\ell,m-1}} = -\underline{1}_{\Lambda \setminus \overline{R}_{\ell,m-1}}$ and $\zeta_{\overline{R}_{\ell,m-1}} = C_{\overline{R}_{\ell,m-1}}$, with $C \in \mathcal{C}$. We remark that $\mathcal{O}_{-\underline{1}}^{(2\lambda-7)}(\eta^{(2\lambda-8)})$ is a set of 2×2 chessboard droplets and $\mathcal{O}_{-\underline{1}}^{(2\lambda-6)}(\eta^{(2\lambda-7)})$ is made of two configurations with a plus spin in the sea of minuses.

We note that Proposition 3.9 implies that the process enters, with high probability in the limit $\beta \rightarrow \infty$, into the set $\mathcal{O}_{-\underline{1}}^{(i+1)}(\eta^{(i)})$, when it exits from the basin of attraction $\mathcal{B}(\eta_i)$ with $\eta_i \in \mathcal{O}_{-\underline{1}}^{(i)}(\eta^{(i-1)})$.

Now, given the $2\lambda - 5$ configurations $\eta_i \in \mathcal{O}_{-\underline{1}}^{(i)}(\eta^{(i-1)})$ for any $i = 0, 1, \dots, 2\lambda - 6$ and the $2\lambda - 6$ integer numbers $t_1 < \dots < t_{2\lambda-6}$ we set $t_0 = 0$ and $t_{2\lambda-5} = t_{2\lambda-6} + 1$, and we say that a path $\omega = \{\omega_{t_0}, \dots, \omega_{t_{2\lambda-6}}\}$ belongs to the set $\mathcal{T}(\eta^{(0)}, \dots, \eta^{(2\lambda-6)}; t_1, \dots, t_{2\lambda-6})$ iff $\omega_{t_i} = \eta^{(i)}$

and $\omega_{t_i}, \dots, \omega_{t_{i+1}-1} \in \overline{\mathcal{B}}(\eta^{(i)})$ for any $i = 0, \dots, 2\lambda - 6$ and $\omega_{t_{2\lambda-5}} = -\underline{1}$. Note that a path in $\mathcal{T}(\eta^{(0)}, \dots, \eta^{(2\lambda-6)}; t_1, \dots, t_{2\lambda-6})$ is one of the “standard” shrinking paths that the system follows when a chessboard droplet $\eta^{(0)} \in \mathcal{O}_{-\underline{1}}^{(0)}$ shrinks. More precisely, given $\eta^{(0)} \in \mathcal{O}_{-\underline{1}}^{(0)}$, we define the tube

$$\mathcal{T}_{\eta^{(0)}} := \bigcup_{i=1}^{2\lambda-6} \bigcup_{\eta^{(i)} \in \mathcal{O}^{(i)}(\eta^{(i-1)})} \bigcup_{t_1 < \dots < t_{2\lambda-6}} \mathcal{T}(\eta^{(0)}, \dots, \eta^{(2\lambda-6)}; t_1, \dots, t_{2\lambda-6}) \quad . \quad (3.30)$$

In other words $\mathcal{T}_{\eta^{(0)}}$ is defined as the set of paths $\omega \in \Theta(\eta^{(0)}, -\underline{1})$ such that there exist $2\lambda - 6$ configurations $\eta^{(i)} \in \mathcal{O}_{-\underline{1}}^{(i)}(\eta^{(i-1)})$ with $i = 1, \dots, 2\lambda - 6$ and $2\lambda - 6$ integer numbers $t_1 < \dots < t_{2\lambda-6}$ such that $\omega \in \mathcal{T}(\eta^{(0)}, \dots, \eta^{(2\lambda-6)}; t_1, \dots, t_{2\lambda-6})$. We state, now, the following Lemma:

Lemma 3.12 *Let $\eta_0 \in \mathcal{O}_{-\underline{1}}$, we have*

$$\mathbb{P}_{\eta^{(0)}} \left(\text{the trajectory } \{\sigma_0, \dots, \sigma_{\tau_{-\underline{1}}}\} \in \mathcal{T}_{\eta^{(0)}} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad .$$

Proof of Lemma 3.12. The Lemma easily follows by applying recursively the Proposition 3.9 and the Markov property. \square

Finally we define the exit tube $\mathcal{T}_{-\underline{1}}$: a path $\omega = \{\omega_0, \dots, \omega_n\}$ is an element of $\mathcal{T}_{-\underline{1}}$ iff there exist $2\lambda - 5$ configurations $\eta^{(i)} \in \mathcal{O}_{-\underline{1}}^{(i)}(\eta^{(i-1)})$ with $i = 0, 1, \dots, 2\lambda - 6$, the integer numbers $t_1 < \dots < t_{2\lambda-6} = n - 1$ and a path $\omega' = \{\omega'_0, \dots, \omega'_n\} \in \mathcal{T}(\eta^{(0)}, \dots, \eta^{(2\lambda-6)}; t_0, \dots, t_{2\lambda-6})$ such that $\omega_i = \omega'_{n-1}$ for any $i = 0, \dots, n$. In other words $\mathcal{T}_{-\underline{1}}$ is the set of paths obtained by time reversing one of the standard shrinking paths associated to the droplets in $\mathcal{O}_{-\underline{1}}$.

Theorem 3.13 *Let σ_t be the process started at $-\underline{1}$, let $\bar{\tau}_{-\underline{1}} := \max\{t < \tau_{\mathcal{S} \setminus \mathcal{A}_{-\underline{1}}} : \sigma_t = -\underline{1}\}$, then*

$$\mathbb{P}_{-\underline{1}} \left(\sigma_{\tau_{\mathcal{S} \setminus \mathcal{A}_{-\underline{1}}}} \in \mathcal{P}_{-\underline{1}}, \text{ the trajectory } \{\sigma_{\bar{\tau}_{-\underline{1}}}, \sigma_{\bar{\tau}_{-\underline{1}}+1}, \dots, \sigma_{\tau_{\mathcal{S} \setminus \mathcal{A}_{-\underline{1}}}-1}\} \in \mathcal{T}_{-\underline{1}} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad .$$

Proof of Theorem 3.13. The theorem is a straightforward consequence of the time-reversing argument (see [OS, S]) and Lemma 3.12. \square

4. The minmax between the metastable and the stable state

The proof of the theorems describing the exit of the system from the metastable state is based on the general lemmata given in the Subsection 3.3. The highly not trivial model dependent part consists in finding the minmax between the metastable and the stable phases. It is clear that new ideas must be used to answer this question in the case of a parallel dynamics with respect to the Glauber case. Indeed, the fact that the system can jump from any configuration to any other, highly complicates the structure of the possible trajectories in the configuration space.

First of all we define a sort of generalized basin of attraction of $-\underline{1}$: let $\mathcal{G}_{-\underline{1}} \subset \mathcal{S}_{-\underline{1}}$ the set

$$\mathcal{G}_{-\underline{1}} := \{\sigma \in \mathcal{S}_{-\underline{1}} : \hat{\sigma} = -\underline{1} \text{ or } \hat{\sigma} \text{ subcritical}\} \quad , \quad (4.1)$$

where $\hat{\sigma}$ subcritical means that $\hat{\sigma}$ is a trap such that there exist $k \geq 1$ rectangles on the dual lattice $R_{\ell_1, m_1}, \dots, R_{\ell_k, m_k}$, with $2 \leq \ell_i \leq m_i \leq L - 2$ for any $i = 1, \dots, k$, there exists $s \in \{0, \dots, k\}$ such that the conditions of point *iii*) in Proposition 3.3 are satisfied (note that in the case $s = 0$ the trap is a stable configuration) and $\ell_i < \lambda$ for any $i = 1, \dots, k$. To fix the ideas: if $\hat{\sigma}$ consisted of a single chessboard rectangle in the sea of minuses, then its shortest side length ℓ should be smaller

than λ . The set \mathcal{G}_{-1} is a sort of “generalized” basin of attraction of the state -1 , in the sense that for any $\sigma \in \mathcal{G}_{-1}$ the process started at σ would visit -1 before exiting \mathcal{G}_{-1} with high probability in the zero temperature limit, namely

$$\mathbb{P}_\sigma \left(\tau_{-1} < \tau_{\mathcal{S} \setminus \mathcal{G}_{-1}} \right) \xrightarrow{\beta \rightarrow \infty} 1. \quad (4.2)$$

In the following Lemma we state the main properties of the basin \mathcal{G}_{-1} : Γ is the minimal energy barrier that must be bypassed to exit \mathcal{G}_{-1} ; a minimal exit path from \mathcal{G}_{-1} reaches $\mathcal{S} \setminus \mathcal{G}_{-1}$ in a protocritical droplet $\eta \in \mathcal{P}_{-1}$.

Lemma 4.1 *Let $\eta \in \mathcal{P}_{-1}$, i) there exists a path $\omega = \{\omega_0 = -1, \dots, \omega_n = \eta\}$ such that $\omega_i \in \mathcal{G}_{-1}$ and $H(\omega_{i-1}, \omega_i) < H(\omega_{n-1}, \omega_n) = H(-1) + \beta\Gamma$ for any $i = 1, \dots, n-1$; ii) there exists a path $\omega' = \{\omega'_0 = \eta, \dots, \omega'_n \in \mathcal{C}\}$ such that $\omega'_i \in \mathcal{S} \setminus \mathcal{G}_{-1}$ and $H(\omega_{i-1}, \omega_i) < H(-1) + \beta\Gamma$ for any $i = 1, \dots, n$. iii) $\Phi(\mathcal{G}_{-1}) = H(-1) + \beta\Gamma$; iv) for all $\sigma \in \mathcal{G}_{-1}$ and $\eta \in \mathcal{S} \setminus \mathcal{G}_{-1}$, $H(\sigma, \eta) = \beta\Gamma + H(-1)$ if and only if $\sigma \in \mathcal{O}_{-1}$ and $\eta \in \pi_{-1}(\sigma)$.*

We postpone the proof of the above lemma to the end of this section. Let us define the set

$$\mathcal{A}_{-1} := \{\eta \in \mathcal{S} : \exists \omega = \{\omega_0 = \eta, \dots, \omega_n = -1\} \text{ such that } \omega_1, \dots, \omega_{n-1} \in \mathcal{G}_{-1} \text{ and } \Phi_\omega < H(-1) + \beta\Gamma\}$$

From Lemma 3.4 and Lemma 4.1 we have that \mathcal{A}_{-1} is a cycle, $\mathcal{A}_{-1} \subset \mathcal{G}_{-1}$, $\Phi(\mathcal{A}_{-1}) = H(-1) + \beta\Gamma$, $U(\mathcal{A}_{-1}) \supset \mathcal{P}_{-1}$ and $\Phi(-1, C) = \beta\Gamma + H(-1)$, with $C \in \mathcal{C}$ (that is $\beta\Gamma + H(-1)$ is the minmax between -1 and C).

Proof of Theorem 3.11. Let σ_t be the process started at -1 . We first try to describe the exit from the basin \mathcal{G}_{-1} by applying Lemma 3.5 and using recurrence in \mathcal{G}_{-1} . We firstly remark that from the reversibility Lemma we have that for each $\varepsilon > 0$

$$\mathbb{P}_{-1} \left(\tau_{\mathcal{S} \setminus \mathcal{G}_{-1}} > e^{\beta\Gamma - \beta\varepsilon} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.3)$$

Now, from item iv) in Lemma 3.5, from the definition of \mathcal{G}_{-1} and from Proposition 3.9 we have that for each $\sigma \in \mathcal{G}_{-1}$ and each $\varepsilon, \delta > 0$

$$\mathbb{P}_\sigma \left(\exists t < e^{\beta\Gamma + \beta\delta}, \sigma_t \in \mathcal{S} \setminus \mathcal{G}_{-1} \right) \geq e^{-\beta\varepsilon} \quad (4.4)$$

For any $\varepsilon > 0$, we set $T(\varepsilon) := \exp\{\beta\Gamma + \beta\varepsilon\}$, $N(\varepsilon) = \lceil \exp\{\beta\varepsilon/2\} \rceil - 1 \in \mathbb{N}$, and consider the intervals $I_k(\varepsilon) := T(\varepsilon) \exp\{-\beta\varepsilon/2\} [k, k+1)$ for any $k = 0, \dots, N(\varepsilon)$. Then

$$\begin{aligned} \mathbb{P}_{-1} \left(\tau_{\mathcal{S} \setminus \mathcal{G}_{-1}} < e^{\beta\Gamma + \beta\varepsilon} \right) &= 1 - \mathbb{P}_{-1} \left(\tau_{\mathcal{S} \setminus \mathcal{G}_{-1}} > e^{\beta\Gamma + \beta\varepsilon} \right) \\ &= 1 - \prod_{k=0}^{N(\varepsilon)} \mathbb{P}_{-1} \left(\sigma_t \in \mathcal{G}_{-1} \text{ for all } t \in I_k(\varepsilon) \right) \\ &= 1 - \sup_{\eta \in \mathcal{G}_{-1}} \left[1 - \mathbb{P}_\eta \left(\exists t < e^{\beta\Gamma + \beta\varepsilon/2}, \sigma_t \in \mathcal{S} \setminus \mathcal{G}_{-1} \right) \right]^{N(\varepsilon)} \\ &\geq 1 - \left[1 - e^{-\beta\varepsilon/2} \right]^{N(\varepsilon)} \end{aligned} \quad (4.5)$$

where we have used (4.4). From (4.5) we get the upper bound on the exit time

$$\mathbb{P}_{-1} \left(\tau_{\mathcal{S} \setminus \mathcal{G}_{-1}} < e^{\beta\Gamma + \beta\varepsilon} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.6)$$

for any $\varepsilon > 0$. Now, by using the reversibility Lemma we have that

$$\mathbb{P}_{-\underline{1}} \left(\sigma_{\tau_{\mathcal{S} \setminus \mathcal{G}_{-\underline{1}}}} \in \mathcal{P}_{-\underline{1}} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.7)$$

and recalling (4.3) and (4.6) we can state that the system prepared in $-\underline{1}$ exits $\mathcal{G}_{-\underline{1}}$ through $\mathcal{P}_{-\underline{1}}$ in a typical time $\exp\{\beta\Gamma\}$. Moreover, $\mathcal{C} \subset \mathcal{S} \setminus \mathcal{G}_{-\underline{1}}$ (the chessboards do not belong to $\mathcal{G}_{-\underline{1}}$) and equation (4.7) imply that $\mathcal{P}_{-\underline{1}}$ is visited before \mathcal{C} , more precisely

$$\mathbb{P}_{-\underline{1}} \left(\tau_{\mathcal{P}_{-\underline{1}}} < \tau_{\mathcal{C}} \right) \xrightarrow{\beta \rightarrow \infty} 1 \quad ,$$

completing the proof of the statement *i*) in Theorem 3.11.

Now, we use the Markov property to restart the system in some configuration of $\mathcal{P}_{-\underline{1}}$. Point *i*) in Proposition 3.9 directly implies point *ii*) in Theorem 3.11. Moreover, point *iii*) in Theorem 3.11 is easily proven by remarking that $\beta\Gamma > 2\beta(2-h)$ and by using Proposition 3.9.

Up to now we have described the jump from $-\underline{1}$ to the chessboards. Now we use the Markov property to restart the system in $C \in \mathcal{C}$ and we prove point *iv*) in Theorem 3.11 by following the same scheme used above. We just sketch the proof: let $\mathcal{G}_{\mathcal{C}} \subset \mathcal{S}_{\mathcal{C}}$ be the set

$$\mathcal{G}_{\mathcal{C}} := \{\sigma \in \mathcal{S}_{\mathcal{C}} : \hat{\sigma} \in \mathcal{C} \text{ or } \hat{\sigma} \text{ subcritical}\} \quad , \quad (4.8)$$

where $\hat{\sigma}$ subcritical means that $\hat{\sigma}$ is a trap such that $k \geq 1$ pairwise non-interacting rectangles $R_{\ell_1, m_1}, \dots, R_{\ell_k, m_k}$ such that $2 \leq \ell_i \leq m_i \leq L-2$ and $\ell_i < \lambda$ for any $i = 1, \dots, k$, $\eta_{\mathcal{R}} = +\underline{1}_{\mathcal{R}}$ and $\eta_{\Lambda \setminus \mathcal{R}} = C_{\Lambda \setminus \mathcal{R}}$ where $\mathcal{R} := \bigcup_{i=1}^k \overline{R}_{\ell_i, m_i}$ and $C \in \mathcal{C}$. To fix the ideas: if $\hat{\sigma}$ consisted of a single plus rectangle in the sea of chessboard, then its shortest side length ℓ should be smaller than λ . Then we state the analogous of Lemma 4.1: let us denote by $\mathcal{Q}_{\mathcal{C}}$ the set of configurations $\eta \in \mathcal{S}$ such that there exists a rectangle $R_{\lambda, \lambda}$ such that $\eta_{\Lambda \setminus \overline{R}_{\lambda, \lambda}} = C_{\Lambda \setminus \overline{R}_{\lambda, \lambda}}$ and $\eta_{\overline{R}_{\lambda, \lambda}} = +\underline{1}_{\overline{R}_{\lambda, \lambda}}$ with $C \in \mathcal{C}$. We denote by $\mathcal{P}_{\mathcal{C}}$ the set of configurations $\eta \in \mathcal{S}$ such that there exist a rectangle $R_{\lambda-1, \lambda}$, a site $x \in \Lambda \setminus \overline{R}_{\lambda-1, \lambda}$, adjacent to one of the two sides of $\overline{R}_{\lambda-1, \lambda}$ of length λ , and $C \in \mathcal{C}$ such that $C(x) = -1$, $\eta_{\Lambda \setminus (\overline{R}_{\lambda, \lambda} \cup \{x\})} = C_{\Lambda \setminus (\overline{R}_{\lambda, \lambda} \cup \{x\})}$, $\eta_{\overline{R}_{\lambda, \lambda}} = +\underline{1}_{\overline{R}_{\lambda, \lambda}}$ and $\eta(x) = +1$.

Lemma 4.2 *Let $C \in \mathcal{C}$, $\eta \in \mathcal{P}_{\mathcal{C}}$, i) there exists a path $\omega = \{\omega_0 = C, \omega_1, \dots, \omega_n = \eta\}$ such that $\omega_i \in \mathcal{G}_{\mathcal{C}}$ and $H(\omega_{i-1}, \omega_i) < H(\omega_{n-1}, \omega_n) = H(C) + \beta\Gamma$ for any $i = 1, \dots, n-1$; ii) there exists a path $\omega' = \{\omega'_0 = \eta, \dots, \omega'_n = +\underline{1}\}$ such that $\omega'_i \in \mathcal{S} \setminus \mathcal{G}_{\mathcal{C}}$ and $H(\omega_{i-1}, \omega_i) < H(C) + \beta\Gamma$ for any $i = 1, \dots, n$. iii) $\Phi(\mathcal{G}_{\mathcal{C}}) = H(C) + \beta\Gamma$.*

As before the statement *iv*) in Theorem 3.11 follows from the Lemmata 4.2, 3.4, 3.5 and point *ii*) in Proposition 3.9. \square

Proof of Lemma 4.1. We start by proving point *i*): let us consider a protocritical droplet $\eta \in \mathcal{P}_{-\underline{1}}$ and the $\lambda \times (\lambda-1)$ chessboard droplet $\eta^{(0)} \in \mathcal{O}_{-\underline{1}}$ such that $\pi_{-\underline{1}}(\eta^{(0)}) = \eta$. First of all we note that: $H(\eta, \eta^{(0)}) = H(-\underline{1}) + \beta\Gamma$.

Now, recall $\mathcal{O}_{-\underline{1}}^{(0)} = \mathcal{O}_{-\underline{1}}$ and consider a sequence of configurations $\eta^{(1)} \in \mathcal{O}_{-\underline{1}}^{(1)}(\eta^{(0)}), \dots, \eta_{2\lambda-6} \in \mathcal{O}_{-\underline{1}}^{(2\lambda-6)}(\eta^{(2\lambda-7)})$. From Proposition 3.9 we have that for any $i = 1, \dots, 2\lambda-7$ the barrier to exit the related basin of attraction is

$$\begin{aligned} \Upsilon(\eta^{(i)}) &\leq H(\eta^{(i)}) + 2\beta h(\lambda-2) < H(\eta^{(0)}) + 2\beta h(\lambda-2) \\ &< H(\eta^{(0)}) + 2\beta(2-h) = H(-\underline{1}) + \beta\Gamma \end{aligned}$$

Note that for $i = 0, 1$ the first inequality is, indeed, an equality.

The above inequalities allow to construct a path $\{\omega_0 = \eta^{(0)}, \dots, \omega_n = -\underline{1}\} \in \mathcal{T}_{\eta^{(0)}}$ connecting $\eta^{(0)}$ to $-\underline{1}$ and such that $H(\omega_i, \omega_i + 1) < H(-\underline{1}) + \beta\Gamma$ for any $i = 0, \dots, n-1$. Finally, we remark that the path $\{\omega'_0 = \omega_n, \omega'_1 = \omega_{n-1}, \dots, \omega'_n = \omega_0, \omega'_{n+1} = \eta\}$ satisfies the properties of point *i*). A similar construction can be repeated for point *ii*).

Now, we come to the main points *iii*) and *iv*): our goal is to prove that $\Phi(\mathcal{G}_{-\underline{1}}) = H(-\underline{1}) + \beta\Gamma$. First of all we notice that point *i*) above implies

$$\Phi(\mathcal{G}_{-\underline{1}}) \leq \beta\Gamma + H(-\underline{1}) \quad ,$$

hence our calculation is reduced to prove the lower bound $\Phi(\mathcal{G}_{-\underline{1}}) \geq \beta\Gamma + H(-\underline{1})$. We have to examine all the paths connecting $\mathcal{G}_{-\underline{1}}$ with $\mathcal{S} \setminus \mathcal{G}_{-\underline{1}}$: such a path $\{\omega_0, \omega_1, \dots, \omega_n\}$ has at least a direct jump from $\mathcal{G}_{-\underline{1}}$ to $\mathcal{S} \setminus \mathcal{G}_{-\underline{1}}$, that is there exists $k \in \{0, \dots, n-1\}$ such that $\omega_k \in \mathcal{G}_{-\underline{1}}$ and $\omega_{k+1} \in \mathcal{S} \setminus \mathcal{G}_{-\underline{1}}$. Hence, for any ω connecting $\mathcal{G}_{-\underline{1}}$ with its exterior we have

$$\Phi_\omega \geq \min_{\sigma \in \mathcal{G}_{-\underline{1}}, \eta \in \mathcal{S} \setminus \mathcal{G}_{-\underline{1}}} H(\sigma, \eta) \quad .$$

Given $\sigma \in \mathcal{G}_{-\underline{1}}$, we consider the configuration $\eta \in \mathcal{S} \setminus \mathcal{G}_{-\underline{1}}$ that can be reached with the smallest energetic cost, namely $\eta \in \mathcal{S} \setminus \mathcal{G}_{-\underline{1}}$ is such that

$$H(\sigma, \eta) = \min_{\zeta \in \mathcal{S} \setminus \mathcal{G}_{-\underline{1}}} H(\sigma, \zeta) \quad (4.9)$$

We have to prove that $H(\sigma, \eta) \geq \beta\Gamma + H(-\underline{1})$ with the equality valid if and only if $\sigma \in \mathcal{O}_{-\underline{1}}$ and $\eta \in \pi_{-\underline{1}}(\sigma)$.

First of all we note that $P_\Lambda(\sigma, \eta) \xrightarrow{\beta \rightarrow \infty} 0$ otherwise we would have $\eta \in \mathcal{B}(\hat{\sigma})$. From Table 1 we get that there exists $x \in \Lambda$ such that

$$\log p_x(\eta(x)|\sigma) = -2\beta(2-h) + o(e^{-\beta c}) \quad (4.10)$$

for some positive constant c . We consider, then, $\zeta = \eta^x$, and we remark that (4.9) implies $\zeta \in \mathcal{G}_{-\underline{1}}$. Note that

$$\log P_\Lambda(\sigma, \eta) = \log P_\Lambda(\sigma, \zeta) - \log p_x(\zeta(x)|\sigma) + \log p_x(\eta(x)|\sigma) \quad (4.11)$$

that, together with (3.4), implies

$$\begin{aligned} H(\sigma, \eta) &= H(\sigma, \zeta) - \log P_\Lambda(\sigma, \eta) + \log P_\Lambda(\sigma, \zeta) \\ &= H(\sigma, \zeta) - \log p_x(\eta(x)|\sigma) + \log p_x(\zeta(x)|\sigma) \\ &\geq H(\zeta) - \log p_x(\eta(x)|\sigma) + \log p_x(\zeta(x)|\sigma) \end{aligned} \quad (4.12)$$

We can characterize $\hat{\zeta}$ as follows: by using Propositions 3.3 and 3.10 we have that there exist $k \geq 1$ rectangles $R_{\ell_1, m_1}, \dots, R_{\ell_k, m_k}$ satisfying the conditions of point *(iii)* in Proposition 3.3, with respect to the configuration $\hat{\zeta}$, and such that $\ell_i < \lambda$ for any $i = 1, \dots, k$.

Let us consider $\zeta' \in \mathcal{S}$ such that $\zeta'_{\cup_{i=1}^k \bar{R}_{\ell_i, m_i}} = -\underline{1}_{\cup_{i=1}^k \bar{R}_{\ell_i, m_i}}$ and $\zeta'_{\Lambda \setminus \cup_{i=1}^k \bar{R}_{\ell_i, m_i}} = \zeta_{\Lambda \setminus \cup_{i=1}^k \bar{R}_{\ell_i, m_i}}$. By recalling that there exist an uphill path joining $\hat{\zeta}$ to ζ it is easy to prove that there exist $s \geq 0$ rectangles $R_{1, m_{k+1}}, \dots, R_{1, m_{k+s}}$, with $m_{k+i} \geq 1$ for all $i = 1, \dots, s$, such that $\zeta'_{\Lambda \setminus \cup_{i=1}^s \bar{R}_{1, m_{k+i}}} = -\underline{1}_{\Lambda \setminus \cup_{i=1}^s \bar{R}_{1, m_{k+i}}}$ and ζ' coincides with a chessboard or $+\underline{1}$ inside $\bar{R}_{1, m_{k+i}}$ for all $i = 1, \dots, s$.

We let b_i^o (resp. b_i^v) the horizontal (resp. vertical) side length of the rectangle R_{ℓ_i, m_i} for any $i = 1, \dots, k+s$. We set $b^o := \sum_{i=1}^{k+s} b_i^o$, $b^v := \sum_{i=1}^{k+s} b_i^v$ and we remark that

$$\sum_{i=1}^{k+s} (\ell_i + m_i) = b^o + b^v \quad . \quad (4.13)$$

We suppose, now, $b^o + b^v \geq 2\lambda$. By a direct evaluation of the energy of the rectangles it is easy to show the bound

$$\begin{aligned} E(\zeta) - E(-\underline{1}) &= [E(\widehat{\zeta}) - E(-\underline{1})] + [E(\zeta) - E(\widehat{\zeta})] \\ &\geq [E(\widehat{\zeta}) - E(-\underline{1})] + [E(\zeta') - E(-\underline{1})] \geq 4(b^o + b^v) - 2h \sum_{i=1}^{k+s} b_i^o b_i^v \end{aligned} \quad (4.14)$$

Where we have used that two of the R_{1,m_i} , with $i = k+1, \dots, k+s$, rectangles can interact iff they are filled with different parity chessboards. Now, by using the subcriticality of $\widehat{\zeta}$, namely by using $\ell_i < \lambda$ for any $i = 1, \dots, k$ (recall $\ell_i = 1$ for all $i = k+1, \dots, k+s$), we have

$$\sum_{i=1}^{k+s} b_i^o b_i^v \leq (\lambda - 1)[b^o + b^v - (\lambda - 1)] \quad (4.15)$$

indeed, let $\ell = \max_{i=1, \dots, k} \ell_i$, we have

$$\sum_{i=1}^{k+s} b_i^o b_i^v = \sum_{i=1}^{k+s} \ell_i m_i \leq \ell \sum_{i=1}^{k+s} m_i \leq \ell[b^o + b^v - \ell] \leq (\lambda - 1)[b^o + b^v - (\lambda - 1)]$$

where, in the last inequality, we have used $b^o + b^v \geq 2\lambda$. Now, recall $\lambda = [2/h] + 1 = 2/h + \varepsilon$ for some $\varepsilon \in (0, 1)$, then the inequality

$$2 - h(\lambda - 1) = 2 - h\left(\frac{2}{h} + \varepsilon - 1\right) = h(1 - \varepsilon) > 0 \quad , \quad (4.16)$$

(4.14), (4.15) and the hypothesis $b^o + b^v \geq 2\lambda$ imply

$$\begin{aligned} E(\zeta) - E(-\underline{1}) &\geq 4(b^o + b^v) - 2h(\lambda - 1)[b^o + b^v - (\lambda - 1)] \\ &= (b^o + b^v)[4 - 2h(\lambda - 1)] + 2h(\lambda - 1)^2 \\ &\geq 2\lambda[4 - 2h(\lambda - 1)] + 2h(\lambda - 1)^2 \\ &= 8\lambda - 2h\lambda^2 + 2h \end{aligned} \quad (4.17)$$

Now, by using (4.10), (4.12) and (4.17) we get that for any $\delta > 0$ there exist β large enough such that

$$H(\sigma, \eta) - H(-\underline{1}) \geq 8\beta\lambda - 2\beta h\lambda^2 + 4\beta - \delta \quad (4.18)$$

Finally, from the equation above it follows, by choosing δ small enough, that $H(\sigma, \eta) > \beta\Gamma + H(-\underline{1})$.

We come, now, to the case $b^o + b^v \leq 2\lambda - 1$. First of all we notice that ζ and η differ for the value of a single spin, this implies that in η there is a single supercritical rectangle. More precisely, there exist $k \geq 1$ rectangles $R_{\ell'_1, m'_1}, \dots, R_{\ell'_{k'}, m'_{k'}}$ satisfying the conditions of point (iii) in Proposition 3.3, with respect to the configuration $\widehat{\eta}$, and such that $\ell'_1 \geq \lambda$ and $\ell'_i < \lambda$ for any $i = 2, \dots, k'$.

Let ℓ^o (resp. ℓ^v) the length of the horizontal (resp. vertical) side of the supercritical rectangle $R_{\ell'_1, m'_1}$. We note that $b^o + b^v \leq 2\lambda - 1$ implies that $\ell^o + \ell^v$ is surely less than 4λ ; one can show, indeed, that $\ell^o + \ell^v$ does not exceed $2\lambda + 4$. Under this condition it is easy to show the bound

$$E(\widehat{\eta}) - E(-\underline{1}) \geq 4(\ell^o + \ell^v) - 2h\ell^o\ell^v \quad , \quad (4.19)$$

indeed the energy of $\widehat{\eta}$ can be bounded from below with the energy of an $\ell^o \times \ell^v$ chessboard droplet in the sea of minuses.

Now, let $C \in \{C^e, C^o\}$ such that $\eta(x) = C(x)$ and consider the collection of rectangles $\mathcal{R} \subset \{R_{\ell_1, m_1}, \dots, R_{\ell_k, m_k}, R_{1, m_{k+1}}, \dots, R_{1, m_{k+s}}\}$, such that for each $R \in \mathcal{R}$ either $\widehat{\zeta}_{\overline{R}} = +\underline{1}_{\overline{R}}$ or the chessboard part of \overline{R} coincides with C . By remarking that \mathcal{R} is a collection of pairwise not interacting rectangles, we can find a positive integer $\Delta \leq 9$ such that if we let

$$V := \overline{R}_{\ell'_1, m'_1} \setminus \bigcup_{R \in \mathcal{R}} \overline{R} \quad \text{and} \quad N := \left| \overline{R}_{\ell'_1, m'_1} \setminus \bigcup_{R \in \mathcal{R}} \overline{R} \right| = |V| \quad (4.20)$$

we get the lower bound

$$E(\eta) - E(-\underline{1}) \geq [E(\widehat{\eta}) - E(-\underline{1})] + 2h(N - \Delta) \geq 4(\ell^o + \ell^v) - 2h\ell^o\ell^v + 2h(N - \Delta) \quad (4.21)$$

where, in the last inequality, we have used (4.19). We remark that N is a lower bound of the number of sites in $R_{\ell'_1, m'_1}$ not belonging to any cluster of ζ that will persist in $\widehat{\eta}$.

We consider, now, the geometrical projection of the rectangles R_{ℓ_i, m_i} , with $i = 1, \dots, k+s$, onto one of the horizontal (resp. vertical) sides of $R_{\ell'_1, m'_1}$. Such a projection is a collection of, maybe not disjoint, segments; we denote with p^o (resp. p^v) the length of the union of these segments. By definition we have $p^o \leq b^o$ (resp. $p^v \leq b^v$). We mention the following interesting bound on N :

$$N \geq \ell^o\ell^v - p^op^v \quad (4.22)$$

Indeed, ℓ^v sites of V are associated to each unit segment of the horizontal side of $R_{\ell'_1, m'_1}$ not belonging to the the projection of the rectangles R_{ℓ_i, m_i} . Moreover, p^o (not already counted) sites of V are associated to each unit segment of the vertical side of $R_{\ell'_1, m'_1}$ not belonging to the the projection of the rectangles R_{ℓ_i, m_i} . Hence,

$$N = |V| \geq (\ell^o - p^o)\ell^v + (\ell^v - p^v)p^o = \ell^o\ell^v - p^op^v \quad .$$

Without loss of generality we can, now, suppose $b^o \leq b^v$. This implies $b^o \leq \lambda - 1$, indeed if it were, by absurdity, $b^o \geq \lambda$, then it would be $b^o + b^v \geq 2\lambda > 2\lambda - 1$. We distinguish among four different situations.

Case 1. $p^o \leq \lambda - 2$. By inserting (4.22) in (4.21) we get

$$\begin{aligned} E(\eta) - E(-\underline{1}) &\geq 4(\ell^o + \ell^v) - 2h\ell^o\ell^v + 2h(\ell^o\ell^v - p^op^v) - 2h\Delta \\ &\geq 4(\ell^o + \ell^v) - 2h(\lambda - 2)\ell^v - 2h\Delta \end{aligned} \quad (4.23)$$

where, in the last inequality, we use $p^o \leq \lambda - 2$ and $p^v \leq \ell^v$. Now, recalling $\ell^v \geq \lambda$, for h small enough we have $\ell^v \geq \Delta$, hence

$$\begin{aligned} E(\eta) - E(-\underline{1}) &\geq 4(\ell^o + \ell^v) - 2h(\lambda - 1)\ell^v = 4\ell^o + 2\ell^v[2 - h(\lambda - 1)] \\ &\geq 4\lambda + 2\lambda[2 - h(\lambda - 1)] = 8\lambda - 2h\lambda^2 + 2h\lambda > \Gamma \end{aligned} \quad (4.24)$$

where we have used (4.16).

Case 2. $p^o = \lambda - 1$ (recall $p^o \leq \lambda - 1$) and $(\ell^o, \ell^v) \neq (\lambda, \lambda)$. As in Case 1 we get

$$E(\eta) - E(-\underline{1}) \geq 4(\ell^o + \ell^v) - 2hp^op^v - 2h\Delta \geq 4(\ell^o + \ell^v) - 2h(\lambda - 1)\lambda - 2h\Delta \quad (4.25)$$

where in the last inequality we have used $p^o = \lambda - 1$ and $p^v \leq \lambda$, indeed

$$p^v \leq b^v \leq 2\lambda - 1 - b^o \leq 2\lambda - 1 - p^o = \lambda$$

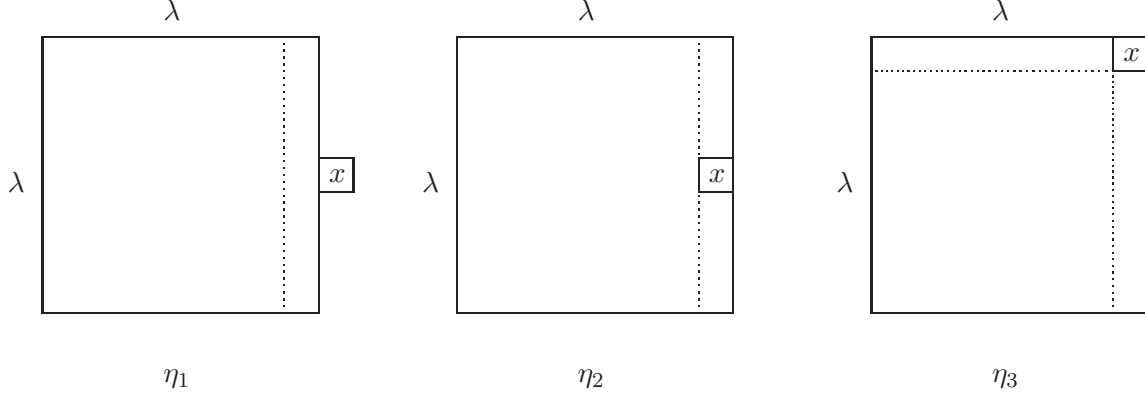


Fig. 4.5: The three possible situations that must be taken into account in Case 4.

Hence, by using $\ell^o + \ell^v \geq 2\lambda + 1$ we get

$$\begin{aligned}
 E(\eta) - E(-\underline{1}) &\geq 4(2\lambda + 1) - 2h(\lambda - 1)\lambda - 2h\Delta \\
 &= 8\lambda - 2h\lambda^2 + 2h(\lambda - 1) + 2h(\lambda - \Delta) + [4 - 2h(\lambda - 1)] \\
 &= \Gamma + 2h(\lambda - \Delta) + [4 - 2h(\lambda - 1)] > \Gamma
 \end{aligned} \tag{4.26}$$

where we have used $4 - 2h(\lambda - 1) > 0$ (see inequality (4.16)) and have chosen h small enough in order to get $\lambda > \Delta$.

Case 3. $p^o = \lambda - 1$, $\ell^o = \ell^v = \lambda$ and $k + s \geq 2$. In this case we have the easy estimate (see [KO] and [NO]) $N \geq 2\lambda$, hence from (4.21) we get

$$E(\eta) - E(-\underline{1}) \geq 8\lambda - 2h\lambda^2 + 2h(2\lambda - \Delta) . \tag{4.27}$$

By choosing h small enough we get $E(\eta) > \Gamma + E(-\underline{1})$.

Case 4. $p^o = \lambda - 1$, $\ell^o = \ell^v = \lambda$ and $k + s = 1$. We have to consider the three configurations η_1 , η_2 and η_3 depicted in Fig. 4.5, where the $\lambda \times (\lambda - 1)$ rectangle is filled with one of the two chessboards and $\eta(x) = +1$. By a direct evaluation of the energy we have that

$$E(\eta_2) = \Gamma - 4h, \quad E(\eta_1) - E(\eta_2) = 4 + 2h > 4h \quad \text{and} \quad E(\eta_3) - E(\eta_2) = 2h(\lambda - 1) > 4h .$$

Now, starting from η_2 the lowest energy jump toward $\mathcal{G}_{-\underline{1}}$ consists in reverting all the spins inside the rectangle R and the plus spin associated to x . The cost of such a jump is $\exp(-4\beta h)$, indeed we have to pay in order to keep the two minuses around x . Finally we have that $H(\sigma, \eta) = \beta\Gamma + H(-\underline{1})$ iff $\sigma \in \mathcal{O}_{-\underline{1}}$ and $\eta \in \pi_{-\underline{1}}(\sigma)$. \square

5. Proof of the Propositions

In this section we prove the Propositions stated throughout the paper: the tools which will be used are those outlined in Subsection 3.3.

The Proposition 3.1 is a straightforward consequence of Lemma 3.2 and the characterization of the local minima of the energy given in [NS1].

Proof of Proposition 3.3. We just give a sketch of the proof. *i)* Let $\sigma \in \mathcal{S}_{+\underline{1}} \setminus \{+\underline{1}\}$. There exists $x \in \Lambda$ with two neighboring sites in the sea of pluses such that $\sigma(x) = -1$, then we will have $T^n \sigma(x) = +1$ for all $n \geq 1$, hence σ is not an element of a stable pair. *ii)* Let $C \in \{C^e, C^o\}$ and $\sigma \in \mathcal{S}_C$. Suppose all the sites of Λ not belonging to the sea of chessboard are occupied by pluses, and suppose these pluses form a single cluster $X \subset \Lambda$. Consider the maximal $Y \subset X$ such that for each $y \in Y$ there exists a 2×2 subset of Y containing y . If Y is not rectangular shaped, then there exists x such that $\sigma(x) = -1$ and at least two among its neighbors belong to Y . Then $T^n \sigma(x) = +1$ for all $n \geq 1$ implies σ is not an element of a stable pair. The proof can be easily generalized to the case with more than a cluster of pluses. *iii)* The proof is similar to the one sketched for the case *ii)*. \square

Proof of Proposition 3.7. Let us consider a rectangle $R_{\ell,m}$ with $2 \leq \ell \leq m \leq L-2$.

Case 1: let $\eta \in \mathcal{S}_{-\underline{1}}$ be the trap such that $\eta_{\Lambda \setminus \overline{R}_{\ell,m}} = -\underline{1}_{\Lambda \setminus \overline{R}_{\ell,m}}$ and $\eta_{\overline{R}_{\ell,m}} = C_{\overline{R}_{\ell,m}}^o$; suppose $\ell < \lambda$. Idea of the proof: we characterize the basin $\mathcal{B}(\eta)$, that is we find $\Upsilon(\eta)$; then we suppose the system prepared in η (namely $\sigma_0 = \eta$) and, by means of Lemmata 3.4 and 3.5, we estimate both $\tau_{\mathcal{S} \setminus \overline{\mathcal{B}}(\sigma)}$ and $\sigma_{\tau_{\mathcal{S} \setminus \overline{\mathcal{B}}(\sigma)}}$.

By using (3.19), $\Upsilon(\eta)$ can be estimated via the construction of the paths in $\Xi(\eta)$. First of all we consider all the possible transitions that can be first steps for a path in $\Xi(\eta)$:

1. the configuration on $\overline{R}_{\ell,m}$ is flipped together with a minus spin in $\Lambda \setminus \overline{R}_{\ell,m}$ adjacent to one of the plus spins in $\overline{R}_{\ell,m}$. A configuration $\eta_1 \in \mathcal{S} \setminus \mathcal{B}(\eta)$ is reached and $H(\eta, \eta_1) - H(\eta) = 2\beta(2 - h) =: \Phi_1$. Hence

$$\Upsilon(\eta) \leq H(\eta) + \Phi_1 = H(\eta) + 2\beta(2 - h) \quad . \quad (5.1)$$

As shown in Fig. 3.3 the unique downhill path starting from η_1 ends in the trap $\hat{\eta}_1$ coincident with a chessboard inside a rectangle $R_{\ell+1,m}$ (or $R_{\ell,m+1}$, depending on which side the protuberance appeared) and with $-\underline{1}$ outside. We notice that the energy of η_1 depends whether the plus protuberance appears in the middle of one side or on the corner, but $H(\eta, \eta_1)$ does not depend on this detail.

2. The configuration on $\overline{R}_{\ell,m}$ is flipped together with a minus spin in $\Lambda \setminus \overline{R}_{\ell,m}$ with four neighboring pluses. A configuration $\eta_2 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_2) - H(\eta) = 2\beta(4 - h) > \Phi_1$: this kind of steps can be neglected.
3. All the spins inside $\overline{R}_{\ell,m}$ are flipped excepted one corner minus (if all the corner are pluses, then this step is considered after a full flip of the configuration inside $\overline{R}_{\ell,m}$). A configuration $\eta_3 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_3) - H(\eta) = 2\beta h$.
4. All the spins inside $\overline{R}_{\ell,m}$ are flipped excepted one minus in the middle (not on the corner) of one of the four sides of the rectangle (if such a spin does not exist, this can happen in the case $\ell = m = 3$, then this step is considered after a full flip of the configuration inside $\overline{R}_{\ell,m}$). A configuration $\eta_4 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_4) - H(\eta) = 2\beta(2 + h) > \Phi_1$: this kind of steps can be neglected.
5. All the spins inside $\overline{R}_{\ell,m}$ are flipped excepted one minus with four nearest neighboring pluses (if such a spin does not exist, this can happen in the case $\ell = m = 3$, then this step is considered after a full flip of the configuration inside $\overline{R}_{\ell,m}$). A configuration $\eta_5 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_5) - H(\eta) = 2\beta(4 + h) > \Phi_1$: this kind of steps can be neglected.

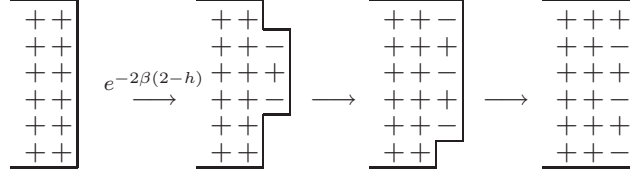


Fig. 5.6: Growth of a plus droplet inside the sea of minuses: appearing of a protuberance.

6. All the spins inside $\bar{R}_{\ell,m}$ are flipped excepted one plus spin. A configuration $\eta_6 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_6) - H(\eta) = 2\beta(4 - h) > \Phi_1$: this kind of steps can be neglected.
7. Two or more events among those listed above are performed simultaneously: the energy cost is smaller than Φ_1 only in the case of a simultaneous persistence of k minus corners of the chessboard. All the others multiple events can be neglected.

From the list above it follows that there exists a path $\omega' = \{\eta, \eta_1\} \in \Xi(\eta)$ consisting of a single step of the first type; so $\Phi_{\omega'} = H(\eta) + \Phi_1 = H(\eta) + 2\beta(2 - h)$. The only paths $\omega \in \Xi(\eta)$ that can compete with ω' are those whose first step is a single or a multiple minus corner persistence. After such a step (see, for instance, the first step in Fig. 3.4) the configuration is a chessboard on a subset of $\bar{R}_{\ell,m}$ obtained by removing some of the four corners of $\bar{R}_{\ell,m}$ and $-\underline{1}$ outside. By a direct inspection it follows that starting from this configuration the possible second steps of our paths are exactly those listed above: no new step enters into the game.

By iterating the above argument, it follows that a path $\omega'' \in \Xi(\eta)$ such that $\Phi_{\omega''} \leq \Phi_{\omega'}$ can be obtained by using only single or multiple minus corner persistences. The best path ω'' is a sequence of $\ell - 1$ minus corner persistences on one of the two sides of the rectangle long ℓ : $\Phi_{\omega''} = H(\eta) + 2\beta h(\ell - 1)$. By comparing $\Phi_{\omega'}$ and $\Phi_{\omega''}$, and recalling that $\ell < \lambda$, one obtains $\Upsilon(\eta) = H(\eta) + 2\beta h(\ell - 1)$. By Lemma 3.4 we obtain $\Phi(\bar{\mathcal{B}}(\eta)) = H(\eta) + 2\beta h(\ell - 1)$ and $U(\bar{\mathcal{B}}(\eta)) = \{\eta''\}$, where η'' is a configuration coincident with a chessboard in a rectangle $R_{\ell,m-1}$ and with $-\underline{1}$ outside. Finally, by applying Lemma 3.5 we can estimate $\tau_{\mathcal{S} \setminus \bar{\mathcal{B}}(\eta)} \sim \exp\{2\beta h(\ell - 1)\}$ and we obtain that with high probability $\sigma_{\mathcal{S} \setminus \bar{\mathcal{B}}(\eta)} = \eta''$. By using the Markov property and by iterating the argument above one completes the proof of part *i*) of Proposition 3.7. The proof of part *ii*) is similar.

Case 2: Let us consider the trap $\eta \in \mathcal{S}_{-\underline{1}}$ such that $\eta_{\Lambda \setminus \bar{R}_{\ell,m}} = -\underline{1}_{\Lambda \setminus \bar{R}_{\ell,m}}$ and $\eta_{\bar{R}_{\ell,m}} = C_{\bar{R}_{\ell,m}}^e$. The proof is the same as in the Case 1.

Case 3: Let us consider the trap $\eta \in \mathcal{S}_{-\underline{1}}$ such that $\eta_{\Lambda \setminus \bar{R}_{\ell,m}} = -\underline{1}_{\Lambda \setminus \bar{R}_{\ell,m}}$ and $\eta_{\bar{R}_{\ell,m}} = +\underline{1}_{\bar{R}_{\ell,m}}$. Again we suppose $\ell < \lambda$. As before we start by listing the transitions that can be first steps for a path in $\Xi(\eta)$:

1. a minus spin adjacent to one of the four sides of the rectangle is flipped. A configuration $\eta_1 \in \mathcal{S} \setminus \mathcal{B}(\eta)$ is reached and $H(\eta, \eta_1) - H(\eta) = 2\beta(2 - h) =: \Phi_1$. We notice that the unique downhill path starting from η_1 ends in a trap η' as in Fig. 5.6.
2. A minus spin at distance greater or equal to two from any site of the rectangle is flipped. A configuration $\eta_2 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_2) - H(\eta) = 2\beta(4 - h) > \Phi_1$: this kind of steps can be neglected.
3. One of the four corners of the rectangle is flipped (a corner is a plus spin with two minuses among its nearest neighbors). A configuration $\eta_3 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_3) - H(\eta) = 2\beta h$.

4. One of the non-corner plus spin on one of the sides of the rectangle is flipped. A configuration $\eta_4 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_4) - H(\eta) = 2\beta(2 + h) > \Phi_1$: this kind of steps can be neglected.
5. One plus spin in the interior of the rectangle is flipped. A configuration $\eta_5 \in \mathcal{B}(\eta)$ is reached such that $H(\eta, \eta_5) - H(\eta) = 2\beta(4 + h) > \Phi_1$: this kind of steps can be neglected.
6. Two or more spins are flipped simultaneously: the energy cost is smaller than Φ_1 only in the case of a simultaneous flip of k corners. All the others many-spin flips can be neglected.

From the list above it follows that there exists a path $\omega' = \{\eta, \eta_1\} \in \Xi(\eta)$ consisting of a single step of the first type; so $\Phi_{\omega'} = H(\eta) + \Phi_1 = H(\eta) + 2\beta(2 - h)$. The only paths $\omega \in \Xi(\eta)$ that can compete with ω' are those whose first step is a single or a multiple corner erosion. Suppose that after the first step of our uphill path the configuration of the system is η_3 . Two more possible transitions must be taken into account in the analysis of the possible second steps:

7. one corner spin is flipped. A configuration $\eta_7 \in \mathcal{B}(\eta)$ is reached such that $H(\eta_3, \eta_7) - H(\eta_3) = 4\beta h$. Indeed we have to take into account that the minus spin with two pluses among its nearest neighbors (the minus at the site flipped at the first step) must persist.
8. The minus spin with two pluses among its nearest neighbors and one of its two adjacent plus spins are simultaneously flipped. A configuration $\eta_8 \in \mathcal{B}(\eta)$ is reached such that $H(\eta_3, \eta_8) - H(\eta_3) = 2\beta h$.

From the third step on no more possible transitions arise, excepted the obvious generalization of 8:

9. a corner plus spin at site x is flipped together with all the spins at sites $y \neq x$ such that $p_y(\eta^y(y)|\eta) \rightarrow 1$ in the limit $\beta \rightarrow \infty$, where η denotes the actual configuration. The energy cost of this transition is $2\beta h$.

We conclude that an estimate of $\Upsilon(\eta)$ smaller than $\Phi_{\omega'} = H(\eta) + 2\beta(2 - h)$ can be obtained only by using an uphill path made of steps of types 3, 7, 8 and 9, or steps in which two or more transitions 3, 7, 8, 9 are performed simultaneously. Consider a path obtained by using these transitions: until on each side of the rectangle there are two nearest neighboring pluses the configuration is still in $\mathcal{B}(\eta)$. Hence, to exit $\mathcal{B}(\eta)$ at least on one of the four sides of the rectangle there must be no pair of nearest neighboring plus spins. It is clear that the path ω'' made of steps 3, 7, 8 and 9, exiting $\mathcal{B}(\eta)$ and with minimal height along the path is the one described in Fig. 5.7: after a first step of type 3 and a second step of type 8, $\ell - 3$ steps of type 9 are performed until the stable pair η'' is reached. The height along this path is $\Phi_{\omega''} = H(\eta) + 2\beta h(\ell - 1)$.

By comparing $\Phi_{\omega'}$ and $\Phi_{\omega''}$ and recalling that $\ell < \lambda$, one obtains $\Upsilon(\eta) = \Phi_{\omega''} = H(\eta) + 2\beta h(\ell - 1)$; by Lemma 3.4 we obtain $\Phi(\overline{\mathcal{B}}(\eta)) = H(\eta) + 2\beta h(\ell - 1)$ and $U(\overline{\mathcal{B}}(\eta)) = \{\eta''\}$. Finally, by applying Lemma 3.5 we can estimate $\tau_{\mathcal{S} \setminus \overline{\mathcal{B}}(\eta)} \sim \exp\{2\beta h(\ell - 1)\}$ and we obtain that with high probability $\sigma_{\tau_{\mathcal{S} \setminus \overline{\mathcal{B}}(\eta)}} = \eta''$. By using the Markov property and the results proven in the Case 1 one completes the proof of part i) of Proposition 3.7. The proof of part ii) is similar. \square

The Propositions 3.8 and 3.10 can be proven via arguments similar to those used in the proof of Proposition 3.7. Proposition 3.9 is a byproduct of the proves of Propositions 3.7 and 3.8.

Acknowledgments

It is a pleasure to express our thanks to J.L. Lebowitz who suggested the problem and to E. Olivieri for many useful discussions and comments. We also thank the CMI of Marseille for its kind hospitality and the European network “Stochastic Analysis and its Applications” ERB-FMRX-CT96-0075 for financial support.

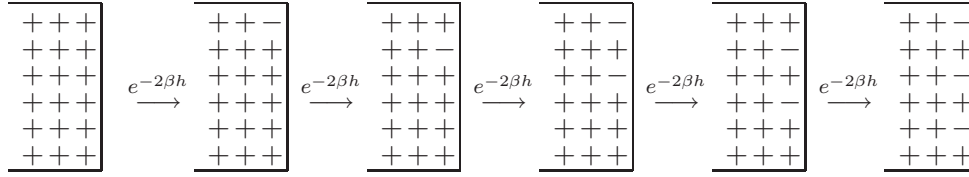


Fig. 5.7: Shrinking of a droplet of pluses inside the sea of minuses.

References

- [BCLS] S. Bigelis, E.N.M. Cirillo, J.L. Lebowitz, E.R. Speer, “Critical droplets in metastable probabilistic cellular automata,” *Phys. Rev. E* **59**, 3935 (1999).
- [C] E.N.M. Cirillo, “A note on the metastability of the Ising model: the alternate updating case,” *J. Stat. Phys.* **106**, 335–390 (2002).
- [CGOV] M. Cassandro, A. Galves, E. Olivieri, M.E. Vares, “Metastable behavior of stochastic dynamics: A pathwise approach,” *J. Stat. Phys.* **35**, 603–634 (1984).
- [D] B. Derrida, “Dynamical phase transition in spin model and automata,” Fundamental problem in Statistical Mechanics VII, H. van Beijeren, Editor, Elsier Science Publisher B.V., (1990).
- [KO] R. Kotecky, E. Olivieri, “Droplet dynamics for asymmetric Ising model,” *J. Stat. Phys.* **70**, 1121–1148 (1993).
- [KV] V. Kozlov, Vasiljev, “Reversible Markov chain with local interactions,” in “Multicomponent random system,” Adv. in Prob. and Rel. Topics, 1980.
- [LMS] J.L. Lebowitz, C. Maes, E. Speer, “Statistical mechanics of probabilistic cellular automata,” *J. Stat. Phys.* **59**, 117–170 (1990); “Probabilistic cellular automata: some statistical mechanics considerations,” in *Lectures in Complex Systems, SFI Studies in the Sciences of Complexity, Lecture Volume II*, ed. E. Jen (Addison Wesley, New York, 1990).
- [NO] F.R. Nardi, E. Olivieri, “Low temperature Stochastic Dynamics for an Ising Model with Alternating Field,” *Markov Proc. and Rel. Fields* **2**, 117–166 (1996).
- [NS1] E.J. Neves, R.H. Schonmann, “Critical Droplets and Metastability for a Glauber Dynamics at Very Low Temperatures,” *Commun. Math. Phys.* **137**, 209–230 (1991).
- [NS2] E.J. Neves, R.H. Schonmann, “Behavior of droplets for a class of Glauber dynamics at very low temperatures,” *Prob. Theor. Rel. Fields* **91**, 331–354 (1992).
- [O] E. Olivieri, private communication.
- [OS] E. Olivieri, E. Scoppola, “Markov chains with exponentially small transition probabilities: First exit problem from a general domain. I. The reversible case,” *J. Stat. Phys.* **79**, 613–647 (1995).

- [PL] O. Penrose, J.L. Lebowitz, “Molecular theory of metastability: An update,” appendix to the reprinted edition of the article “Toward a rigorous molecular theory of metastability,” by the same authors, in *Fluctuation Phenomena* (second edition), eds. E.W. Montroll, J.L. Lebowitz (North-Holland Physics Publishing, Amsterdam, 1987).
- [R] P. Rujan, “Cellular Automata and Statistical Mechanical Models,” *J. Stat. Phys* **49**, 139–222 (1987); A. Georges, P. Le Doussal, “From Equilibrium Spin Models to Probabilistic Cellular Automata,” *J. Stat. Phys.* **54**, 1989.
- [S] R.H. Schonmann, “The pattern of escape from metastability of a stochastic Ising model,” *Commun. Math. Phys.* **147**, 231–240 (1992).
- [St] O.N. Stavskaja, “Gibbs invariant measures for Markov chains on finite lattices with local interactions,” *Math. USSR Sobrnik* **21**, 395–411 (1973). A.L. Toom, N.B. Vasilyev, O.N. Stavskaja, L.G. Mitjushin, G.L. Kurdomov, S.A. Pirogov, “Discrete Local Markov Systems.” Preprint 1989.
- [V] Vasiljev, “Bernoulli and Markov stationary measures in discrete local interactions,” *Lect. Notes in Math.* 653, 1978.